FINDING THE BINDING ENERGY FOR A DEUTERON IMMERSED IN A VAPOR OF NUCLEONS USING GAUSSIAN POTENTIAL AND THE VARIATIONAL PRINCIPLE

إيجاد طاقة الربط للديوترون المغموس في بخار من النيوكليونات باستخدام جهد غاوسي و مبدأ التباين

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الإهداء

إلى من علموني العطاء بلا مقابل، إلى من آثروني على أنفسهم، إلى من كان دعاؤهم سر نجاحي

أمي و أبي

إلى من هم أقرب إلي من روحي، إلى من شاركوني لحظات الحياة حلوها و مرها، إلى سندي و قوتي

إخوتي

إلى رفيق دربي و بلسم روحي، إلى من رافقني مشواري خطوة بخطوة

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إلى من صبر علي و لم يبخل علي بعلمه،

دكتور هنري

إلى كل من ساهم في وصولي إلى هذه المرحلة، إلى كل من علمني حرفا

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إليكم جميعا أهدي بحثي هذا

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ملخص

إن المادة النووية قليلة الكثافة و الموجودة في درجات حرارة منخفضة تتكون من البروتونات و النيوترونات بالإضافة إلى العديد من الأنوية الخفيفة كالديوترون، والذي هو موضوع دراستنا هذه. خصائص هذه الأنوية تتأثر بوجود الوسط المحيط من الأنوية الأخرى و النيوكليونات الحرة، كما أن طاقة الربط لها أقل مما هي عليه في الأنوية العادية الموجودة في الفراغ و غير محاطة بهذه المادة النووية، وطاقة الربط تقل بازدياد كثافة هذه المادة حتى تصل كثافة المادة النووية إلى ما يسمى بكثافة "مُتْ"، وهي الكثافة التي يتفكك عندها الديوترون إلى مكوناته الأساسية، البروتون و النيوترون.

و قد قمنا في بحثنا هذا بدراسة تأثير المادة النووية على طاقة الربط للديوترون آخذين بعين الاعتبار تأثير باولي بالإضافة إلى تأثير كمية التحرك للديوترون حيث استخدمنا الجهد الغاوسي. كما أخذنا بعين الاعتبار تغير اقتران الدالة و اعتماده على كثافة المادة النووية المحيطة و ذلك باستخدام مبدأ التباين.

ABSTRACT

Nuclear clusters such as deuterons exist in nuclear matter at low density in addition to the unbound protons and neutrons. The properties of these clusters including their binding energies are affected by the surrounding vapor. Their binding energies are less than the corresponding regular nuclei in vacuum and these binding energies decrease as the vapor density increases. When the density reaches the Mott density, at which the deuteron dissolves and becomes unbound, the Mott transition occurs, the deuteron dissolves and becomes unbound. In our research we study the binding energy of a deuteron immersed in a vapor of nucleons as a function of the nuclear number density by taking into account the Pauli blocking shift and the center of mass (CM) momentum. We also include the change in the internal wavefunction in the presence of the vapor by using the Variational Principle.

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CHAPTER 1

INTRODUCTION

The properties of the nucleus and its physical behavior can be studied using different nuclear models. One of these models is the liquid drop model which describes finite nuclei. In this model the nucleus is considered as an incompressible liquid droplet with a sharp boundary and uniform density with the existence of the coulomb energy associated with the protons. The particles at the surface of the nucleus interact less than the particles in the interior of the nucleus and this is similar to the surface tension of the liquid.

We know that the volume of a droplet of a liquid increases with increasing the number of atoms in it. The same happens with the nucleus in the liquid drop model, the volume of the nucleus is proportional to the number of nucleons contained in it [1-2].

The features of the liquid drop model lead to the Bethe - Weizsäcker formula for the binding energy of a finite nucleus of Z protons and $N = (A - Z)$ neutrons [3]

$$B(Z, A) = \alpha_v A - \alpha_s A^{\frac{2}{3}} - \alpha_c \frac{Z(Z-1)}{A^{\frac{1}{3}}} - \alpha_{symm} \frac{(A-2Z)^2}{A} + \Delta E_{pair} \tag{1.1}$$

The parameters in the equation (1.1) above are [3]

α_v: volume term	16 MeV
α_s: surface term	17 MeV
α_c: coulomb term	0.6 MeV
α_{symm}: symmetry term	25 MeV

Table 1.1: The parameters in Bethe -Weizsäcker formula

$$\Delta E_{pair} = \begin{cases} \delta \quad \text{for even} - \text{even nuclei} \\ 0 \quad \text{for odd} - \text{even nuclei} \\ -\delta \ \text{for odd} - \text{odd nuclei} \end{cases} \text{, Where } \delta = \frac{25}{A} \text{MeV}$$

The first term is the volume term, the volume V of the nucleus is proportional to R^3

$$V = \frac{4}{3}\pi R^3 \tag{1.2}$$

where R is the radius of the nucleus is given by the equation

$$R = r_0 A^{1/3} \tag{1.3}$$

with $r_0 = 1.2\ fm$ and A is the number of nucleons.

By substituting (1.3) in (1.2) we can notice that the volume is proportional to the nucleon number A.

The decrease in the binding energy due to the number of nucleons on the surface is given by the second term. The third term results from the repulsive coulomb interaction between all protons in the nucleus. Each pair of protons contributes to the Coulomb term equally, the number of protons is Z, so the number of proton pairs is Z(Z-1)/2. The fourth term which includes the quadratic dependence on $(A - 2Z)$ is very important for light nuclei, it expresses the fact that nuclei with $Z \approx \frac{A}{2}$ are expected to be more stable because the protons and the neutrons will occupy the lowest energy state. This term is called the symmetry term because it tends to make the nucleus symmetric in protons and neutrons $Z \approx N$. The last term in equation (1.1) ΔE_{pair} is the pairing energy, it takes into account the tendency of like nucleons to form pairs in order to decrease the energy of the nuclear system. If both N and Z are odd the nucleus will usually be unstable and one of the odd neutrons will be transformed into a proton or vice versa by beta decay to form a pair, while if both N and Z are even the nucleus will be more tightly bound than if they are odd integers. But if the total number of nucleons is odd (N or Z is odd) the pairing energy term has no contribution [1, 4] and this is clear from the values of the parameter ΔE_{pair}.

If we divide equation (1.1) by A we get the binding energy per nucleon for finite nuclei

$$\frac{B(Z,N)}{A} = \alpha_v - \frac{\alpha_s}{A^{1/3}} - \alpha_c \frac{Z(Z-1)}{A^{4/3}} - \alpha_{symm} \frac{(N-Z)^2}{A^2} + \frac{\Delta E_{pair}}{A}$$

In [4] it is shown that the value of binding energy per nucleon is about 8 MeV which is relatively constant except for light nuclei with $A < 20$.

Another characteristic for finite nuclei is their density (ρ) which is about:

$\rho \approx 0.14\, nucleon/fm^3$ in the core of the nucleus but this quantity decreases gradually to zero with radial distance in the surface region. We can notice this from the figure (1.1) below which shows the charge density (protons density) for oxygen (^{16}O), nickel (^{58}Ni), and lead (^{208}Pb) [4].

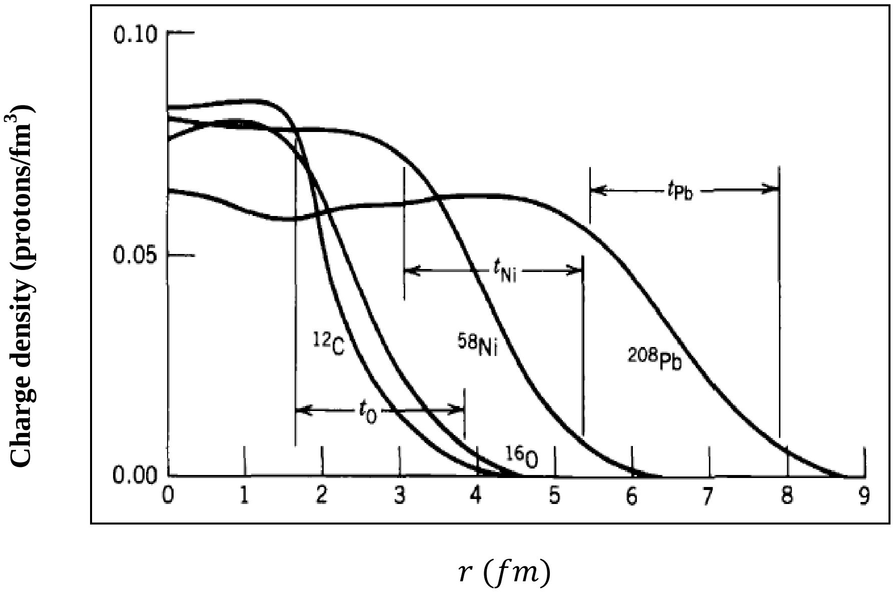

Figure 1.1: Charge density for several nuclei as a function of radial distance r

In figure (1.1), if we look at heavy nuclei like lead (Pb^{208}), it is clear that the charge density is roughly constant, about 0.07 protons/fm^3. Since we can consider the neutron and proton densities are the same, the nucleon density for lead will be approximately $0.14\, nucleons/fm^3$. But this value drops slowly to zero over a distance t of about 2.3 fm. The parameter t is called the skin thickness parameter and it represents the distance at which the charge density of the nucleus drops to 10% of its central value. The value of t is approximately the same for all nuclei (2.3 fm), it does not depend on the size of the nucleus.

For ideal system of interacting nucleons with uniform density we use nuclear matter (infinite nuclear matter) instead of finite nuclei. In nuclear matter the number of nucleons (A) is infinite and the Coulomb force is switched off. For more simplification, we can assume that nuclear matter is symmetric so $N = Z$.

The binding energy per nucleon for nuclear matter and the saturation density are fundamental constants of nature [5]. These constants can be determined from two different sources, the Bethe -Weizsäcker formula and electron scattering on finite nuclei.

To find the binding energy per nucleon for symmetric nuclear matter we can use the Bethe -Weizsäcker formula. Since A is infinite and the Coulomb force is switched off and $N = Z$ all terms in the formula vanish except the volume term. So the binding energy per nucleon for nuclear matter is

$$\frac{B(Z,N)}{A} = \alpha_v$$

From the table (1.1) above, the value of α_v is $16\, MeV$ and hence the binding energy per nucleon for nuclear matter is $16\, MeV$ which differs from the value of 8 MeV for finite nuclei.

The saturation density of nuclear matter can be obtained from electron scattering on finite nuclei and its value is about [3]

$$\rho_0 = 0.16 \pm 0.02\, nucleons/fm^3$$

This density is uniform throughout the nuclear volume and it is the central density of heavy nuclei. Due to the absence of a surface region in nuclear matter, the saturation density differs from the density for finite nuclei which is approximately

$$\rho \approx 0.14\, nucleon/fm^3$$

Nuclear matter exists in two phases; it can exist in a dilute gaseous phase or it can exist in a liquid phase with closely interacting nucleons. The transition between these phases can occur; the nuclear matter transfers from the liquid phase to the gaseous one where the average inter-particle distance is much larger than the range of the inter-particle interaction [6]. The liquid-gas phase transition occurs in general in systems with short-range repulsive and longer-range attractive forces [7]. The transition between the phases occurs at all temperatures below a critical temperature. At these temperatures, the two distinct phases coexist; matter inside the nucleus is in the liquid phase while the surrounding external matter is in the gaseous phase [8]. The behavior of the phase diagram of nuclear matter isotherms introduces theoretical evidence of the coexistence of these phases. These isotherms are very similar to those obtained from a Van der Waal equation of state [9]. At the critical temperature, the distinction between these phases disappears. Above this temperature only the gaseous phase can exist. In [6] Jaqaman investigated the occurrence of liquid-gas phase transition in finite nuclei and he found that there is a reduction in the critical temperature as compared to the infinite nuclear matter.

At very low densities, less than one tenth of the nuclear saturation density ρ_0, nuclear matter exists in a dilute gaseous phase while at higher densities it exists in liquid phase with closely-interacting nucleons [10].

Much below the saturation density, at one hundredth or one thousandth of saturation density, clusters are formed in the gaseous phase to minimize the energy of the system [11-12]. The binding energy for these clusters depends on the density of the surrounding vapor. As the density of the surrounding vapor approaches zero the binding energy for the light clusters becomes very close to the binding energy for isolated nuclei with the same number of protons and neutrons. The formation of light clusters, up to the alpha particle, in nuclear matter at finite temperature and very low density was investigated by Typel *et. al* [11]. They also studied the dissolution of these clusters due to medium effect using the microscopic quantum statistical (QS) approach and a generalized relativistic mean field (RMF) model.

In 2006, the formation of clusters in low density nuclear matter composed of protons, neutrons, and alpha particles was studied in [12] using the virial expansion.

Beyer *et. al*. [13] found that for the clusters immersed in a vapor of nucleons, the Pauli blocking has to be taken into consideration in addition to the self energy shift. The Pauli blocking effect is related to the Pauli Exclusion Principle which prevents two identical nucleons from occupying the same quantum state. For clusters immersed in a vapor of nucleons, Pauli blocking effect is caused by the indistinguishability between the nucleons inside the clusters and the free nucleons in the surrounding vapor. As a result, the total wavefunction involving the nucleons inside and outside the nucleus should be antisymmetric [1, 3]. By increasing the density of the nucleons in the vapor the binding energy of the cluster is decreased because of Pauli blocking [13]. When the density reaches the Mott density, at which the cluster has zero binding energy, the Mott transition occurs, the cluster dissolves and becomes unbound and this depends on the momentum of its center of mass and on the temperature.

In our work, we are interested in studying the medium effect on the deuteron which is the simplest bound state of nucleons since it consists of a proton and a neutron only [1]. We will see what will happen for the binding energy of the deuteron when it is immersed in a vapor of nucleons and what is the effect of Pauli blocking on the Mott density.

In studies [11, 14], the formula for Pauli blocking was found indirectly by calculating the Pauli blocking energy shift at zero CM momentum for the deuteron, and then some approximations and fits were used such as angular averaging.

In a recent work, Abdul-Rahman, Alstaty and Jaqaman [10] used the methods of quantum and statistical mechanics to calculate the binding energy for the deuteron in low density nuclear matter and to get a formula for the Pauli blocking shift that explicitly depends on the deuteron CM momentum with no fits. They found the Mott densities at different temperatures, for the two cases of zero and nonzero CM momenta for the deuterons and compared them with values obtained by Typel *et al* [11]. They found that the Mott densities they got at low temperatures are approximately twice larger than the densities obtained by Typel *et al* and at high temperatures they are three times those of Typel *et al.* In [10] it was assumed that the internal wavefunction of the deuteron is not affected by the presence of the vapor. What we want to do in the present work is to include the change in the internal wavefunction in the presence of the vapor by using the variational principle.

We will start this thesis by explaining the main properties of the deuteron in addition to the main characteristics of the nucleon-nucleon interaction. We will do this in chapter 2.

In chapter 3, we will construct the wavefunction for the deuteron-nucleon system where the deuteron and the free nucleon are confined in a small box, then we will find the energy expectation value for the deuteron-nucleon system and write the binding energy formula including all nucleons in the surrounding vapor.

The expectation value for the quantities which include the CM momenta is calculated in chapter 4 at high temperatures and at absolute zero temperature. The depth and the range for the Gaussian potential are determined in chapter 5. In the same chapter, the Gaussian wavefunction parameter (η) is evaluated for the case where the wavefunction is not affected by the vapor density, then the results for binding energy and Mott density are compared with the results in [10]. In chapter 6, the wavefunction is allowed to vary with the vapor density, and the variational principle is used to evaluate the value of η to find the Mott density for the deuteron at different temperatures. At the end, in chapter 7, we will summarize and discuss our results

Chapter 2

THE DEUTERON

In this chapter, we will discuss the main properties of the deuteron and some characteristics of the nucleon-nucleon interaction.

2.1 PROPERTIES OF THE DEUTERON

The deuteron is the simplest two-nucleon bound system; it consists of a proton and a neutron held together by attractive forces. Due to this simplicity, the deuteron gives us an ideal system for studying the nucleon-nucleon interaction.

The binding energy for any nucleus is defined as the negative of the difference between the nuclear mass and the sum of the masses of the constituents

$$B_0 = -(M_n - zm_p - (A - Z)m_n)c^2 \tag{2.1}$$

Where M_n is the nuclear mass, m_p is the proton's mass, m_n is the neutron's mass, Z is the number of protons and A is the mass number. For the deuteron, the binding energy is 2.225 MeV which is relatively small compared with typical nuclei for which the average binding energy per nucleon is about 8 MeV. Because of the small binding energy, the deuteron is a weakly bound system and it has no excited state [1, 4, 15]. The main ground state properties of the deuteron are listed in the table (2.1) below [3].

Ground state property	Value
Binding energy, B_0	$2.22457312(22)$ MeV
Spin and parity, J^{π}	1^{+}
Isospin, T	0
Magnetic dipole moment, μ_d	$0.857438230(24)\ \mu_N$
Electric quadrupole moment, Q_d	$0.28590(30)\ e.\mathrm{fm}^2$
Matter radius, r_d	$1.963(4)$ fm

Table 2.1: Ground state properties of the deuteron

The total angular momentum **J** of the deuteron is the sum of 3 terms which are the individual spin of the proton $\mathbf{s_p}$ and of the neutron $\mathbf{s_n}$, each with spin 1/2 because they are fermions, and the orbital angular momentum **L** of the nucleons as a result of their motion about their center of mass. **L** can take the values 0, 1, 2, 3, ..., which are usually called S, P, D, F, ... states. So the total angular momentum **J** of the deuteron can be written as

J=S + L

Where $\mathbf{S} = \mathbf{s_p} + \mathbf{s_n}$ is the total spin [4]. The total angular momentum of the deuteron is **J**=1[1]. The proton and neutron spins can be either parallel for a total spin of 1(**S=1**) which is the triplet state, or antiparallel for a total spin of 0 (**S=0**) which is the singlet state. To get **J=1** there are four combinations between **S** and **L**.

a) **S=1** with **L=0**,

b) **S=0** with **L=1**,

c) **S=1** with **L=1**,

d) **S=1**with **L=2**,

But which one of these combinations is accepted? This can be determined after we know the parity of the deuteron.

The parity determines the behavior of the wavefunction when $\vec{r} \rightarrow -\vec{r}$. By studying the reactions involving deuterons and the properties of the photon emitted during the formation of the deuterons, it was found that the parity of the deuteron is positive. But the wavefunction of the deuteron is the product of the intrinsic wavefunction of the proton, the intrinsic wavefunction of the neutron and the orbital wavefunction for the relative motion of the proton and the neutron. The parity of intrinsic wavefunctions of the proton and the neutron are the same because they are two different states of the nucleon so the product of their intrinsic wavefunction is positive. From [16] we know that the parity associated with orbital motion is determined by the factor $(-1)^L$ because the angular dependence in the wavefunction of the deuteron is given by the spherical harmonic Y_{Lm} and when the parity operation is applied to Y_{Lm}, $Y_{Lm}(\pi-\theta, \varphi+\pi) = (-1)^L Y_{Lm}(\theta, \varphi)$ it gives a phase $(-1)^L$. So the positive parity will be for **L=0** which is the S state and for **L=2** which is the D state. Now we can eliminate the combinations which include **L=1**. It is clear now that it is impossible for the deuteron to be in the singlet state where s_p and s_n antiparallel with a total spin 0 (**S=0**), but it exists in the triplet state (**S=1**) [4].

The two possibilities of **L** indicate that both the triplet S state with **L=0** and total spin **S=1** ($^3S_1$) and the triplet D state with **L=2** and total spin **S=1** ($^3D_1$) components appear in the ground state wave function of the deuteron

$$\psi_d = C_S\,{}^3S_1 + C_D\,{}^3D_1.$$

The triplet S state ($^3S_1$) is spherically symmetric since it doesn't have angular momentum, but the existence of triplet D state ($^3D_1$) breaks this symmetry. The coefficients C_S and C_D can be determined by the electric quadrupole and the magnetic dipole moments beside the normalization condition

$${C_S}^2 + {C_D}^2 = 1$$

The electric quadrupole moment measures the departure (the deviation) from a spherical charge distribution of a nucleus. For pure $^3S_1$ state the electric quadrupole moment is zero, but for the deuteron, the electric quadrupole moment is a positive quantity ($Q_d = 0.28590\ e.fm^2$) which is an evidence for the presence of the $^3D_1$ in the ground state for the deuteron.

On the other hand, the magnetic dipole moment of the deuteron also indicates that the ground state of the deuteron is a mixture of $^3S_1$and $^3D_1$. This can be verified by calculating the magnetic dipole moment for the ground state by assuming that the deuteron has no orbital angular momentum (L=0, S state), and so the magnetic dipole moment is the sum of the magnetic dipole moments for the free proton and free neutron, where the magnetic dipole moment for the proton is $\mu_p = 2.7928456\,\mu_N$, while for the neutron is $\mu_n = -1.9130418\,\mu_N$ with $\mu_p + \mu_n = 0.8798038\,\mu_N$, but the experimental value of the deuteron magnetic dipole moment is $\mu_d = 0.8574376\,\mu_N$ [4]. The small difference between μ_d and $\mu_p + \mu_n$ indicates that the ground state of the deuteron is not a pure ${}^3S_1$ state, the ${}^3D_1$ state has a small but nonzero contribution to the deuteron's ground state.

Using the experimental values for the electric quadrupole moment and the magnetic dipole moment of the deuteron beside the normalization condition it is found that the probability of the deuteron to be in the ${}^3S_1$ state (${C_S}^2$) is about 96% and the probability to be in the ${}^3D_1$ state (${C_D}^2$) is about 4%. This means that the ${}^3S_1$ state is dominant in the ground state of the deuteron, while the ${}^3D_1$ state has only a small contribution. Because of the presence of ${}^3D_1$ state we conclude that the nucleon-nucleon interaction is not spherically symmetric and hence it is not a purely central potential. This combination between ${}^3S_1$ state and ${}^3D_1$ state has a very important role in the study of the properties of the nucleon-nucleon interaction.

2.2 NUCLEON-NUCLEON INTERACTION

As we mentioned at the beginning of this chapter, the deuteron gives us an ideal system to study the nucleon-nucleon interaction, which plays a vital role in understanding the nuclear force. In principle, the existence of stable nuclei implies that the net nucleon-nucleon force must be attractive and much greater than the Coulomb force. Of course this attractive force is not electric because the neutrons have no charge [2]. So what is the nature of these forces? How do they depend on the distance between nucleons and on their spins?

Nucleons (protons and neutrons) are not elementary particles, they are the bound states of three fermions with spin 1/2 which are called quarks.

The proton consists of two up quarks and one down quark (proton=uud), whereas the neutron consists of two down quarks and one up quark (neutron=udd) [17]. The difference between their masses is relatively small. It was found that the mass of the proton is 938.272 MeV/c^2 while the mass of the neutron is 939.566 MeV/c^2. They differ only by 0.1% [3].

The charge of the up quark is + 2/3 e, where e is the magnitude of the electric charge of the electron which is equal to $1.60217733(49)\times 10^{-19}$ C, and the charge of the down quark is -1/3 e. From this, it is easy to conclude that the charge of the proton is +1e while the neutron is neutral [3].

Both proton and neutron have spin 1/2 so they are fermions, particles that obey Fermi-Dirac statistic. When nucleons interact with each other the Pauli exclusion principle must be applied and the total wavefunction should be antisymmetric [3].

The proton and the neutron can be considered as two states of the same particle. They are similar in most of their properties; both have spin 1/2 and their masses are very close to each other, they differ only by about 0.1%. The main difference between the proton and the neutron is in their electromagnetic properties. But if we are dealing with the strong interaction we cannot distinguish between them, they will be considered as 2 states of the same particle.

To distinguish between them we need a new label, this label (operator) is the isospin. The value of the isospin for the nucleon is t=1/2. The proton and the neutron are two different states of the nucleon, they differ by the third component of isospin operator t_0. While the neutron has $t_0= +1/2$ the proton has $t_0= -1/2$.

In addition to the protons and neutrons, there exist many particles that consist of quarks and antiquarks. The most important particles for nuclear physics are the three pions (π^+,π^0,π^-) [1]. The $\pi^+$is made up of an up quark and an anti-down quark ($\pi^+ = u\bar{d}$), while the π^-consists of a down quark and an anti-up quark ($\pi^- = d\bar{u}$), and π^0 is composed of a mixture of up, anti-up, down, and anti-down quarks ($\pi^0 = \frac{u\bar{u}-d\bar{\bar{d}}}{\sqrt{2}}$)[18-19].

The oldest attempt to explain the nature of the nuclear force was proposed by Yukawa in 1934. He supposed that nucleons are attracted together due to exchange of quanta of nonzero mass, which were later identified as the pions (π^+,π^0,π^-) [20].

If we are interested in the low-energy region where the nucleons hardly get excited internally, we can treat the nucleons as inert, structureless elementary particles, and we can understand many of the properties of the multi-nucleon systems by the nucleon-nucleon interactions.

The main features of the nuclear force can be summarized as follows:

1. The nuclear force has a finite range. This property can be deduced by noticing that the binding energy per nucleon (8 MeV) [3] and the densities of heavy nuclei ($0.16 \pm 0.02\, nucleons/fm^3$) [3-4] are nearly constant. If the range of the nuclear force were infinite then both quantities will increase by increasing the number of nucleons.

2. The nucleus contains protons and neutrons. In spite of the existence of the repulsive force between protons the nucleus is stable. This stability is caused by

the existence of the nuclear force which is a short-range attractive force; the range of this force is about 1.4 fm which is in the order of nuclear radius. The reason that nuclear force has a short range comes from the exchange of pions between nucleons inside the nucleus which was proposed by Yukawa. By considering that when two nucleons interact they exchange the pion with mass $m = 140\, MeV/c^2$ [21], then the energy violation ΔE of this particle is approximately mc^2, where c is the speed of light.

From the uncertainty principle the pion can exist for a time interval

$$\Delta t \approx \frac{\hbar}{\Delta E} = \frac{\hbar}{mc^2} \qquad (2.2)$$

Where $\hbar$ is the reduced Planck constant. If one considers, at most, that the particle moves with the speed of light c, then, during the time interval Δt it can travel, at most, a distance $c\Delta t$. If we take this distance to be the range of nuclear force then

the range of the nuclear force = $c\Delta t \approx \frac{\hbar c}{mc^2} \approx 1.4\, fm$

but the nuclear force rapidly decreases to zero beyond this distance. At short distances (less than 0.6 fm) [22], the interaction between nucleons becomes repulsive due to the exchange of other mesons like $\omega = \left(\frac{u\bar{u}+d\bar{d}}{\sqrt{2}}\right)$ and $\rho = \left(\frac{u\bar{u}+d\bar{d}-s\bar{s}}{\sqrt{6}}\right)$ where s is the strange quark [19, 22-23]. This repulsion protects the nuclei from collapsing and it is responsible for the size of the nuclei.

3. The nucleon-nucleon interaction is strongly spin dependent. This property is mainly predicted from the fact that only the triplet state exists in the deuteron. As

a result, an additional term which depends on the spins of the two nucleons, $\vec{s}_1$and $\vec{s}_2$, must be added to the central potential. Experiments indicate that nuclear forces satisfy certain symmetries such as invariance under parity ($\vec{r} \to -\vec{r}$) and time reversal ($t \to -t$),so the spin term must also satisfy the parity and the time reversal invariances.

4. The nucleon-nucleon force has a non-central or tensor component. This part of the force does not conserve orbital angular momentum, which is a constant of motion under central forces [4]. We mentioned above that there is a combination between ${}^3S_1$ state and ${}^3D_1$ state where ${}^3S_1$ state is dominant while ${}^3D_1$ state has a small contribution. This indicates that a small non-central force component must be added to the central dominant force between two nucleons. As the only reference direction for a nucleon is its spin, the tensor force term depends mainly on the separation position vector $\vec{r}$ and the spin of the nucleon $\vec{s}$. Thus there are only two terms relating $\vec{r}$ and $\vec{s}$ with each other that can contribute, $(\vec{s}.\vec{r})$ or $(\vec{s} \times \vec{r})$. For two nucleons with spins $\vec{s}_1$and $\vec{s}_2$, the tensor potential depends on the terms $(\vec{s}_1.\vec{r})(\vec{s}_2.\vec{r})$ or $(\vec{s}_1 \times \vec{r}).(\vec{s}_2 \times \vec{r})$. But we can write the second term in terms of the first with an extra term $r^2(\vec{s}_1.\vec{s}_2)$ using the properties of dot and cross products. $(\vec{s}_1 \times \vec{r}).(\vec{s}_2 \times \vec{r}) = \vec{s}_2.\left(\vec{r} \times (\vec{s}_1 \times \vec{r})\right) = \vec{s}_2.\left(\vec{s}_1(\vec{r}.\vec{r}) - \vec{r}(\vec{r}.\vec{s}_1)\right)$

   $= r^2\left(\vec{s}_1.\vec{s}_2\right) - (\vec{s}_1.\vec{r})(\vec{s}_2.\vec{r}))$ . .

   As a result, the tensor character of the nucleon-nucleon interaction can be written as $S_{12} = \frac{3(\vec{s}_1.\vec{r})(\vec{s}_2.\vec{r})}{r^2} - \vec{s}_1.\vec{s}_2$. If we consider infinite nuclear matter with many

nucleons, the tensor term can be ignored because the average over all angles is zero.

5. The nucleon-nucleon interaction is assumed to be charge independent [4, 20]. This means that the proton-proton, the neutron-neutron, and the proton-neutron interactions are identical although there is a very small difference (of order 1%) between the potentials of the proton-neutron interaction from one side, and the proton-proton and neutron-neutron interactions from the other side [4].

Determining the correct potential between the nuclear particles is one of the important questions in nuclear physics. The proper potential helps us to calculate the main properties of the nucleus and compare the results with experimental data.

The forces between nucleons have a very short range which is about 1.4 fm, they are very strong within this range but rapidly decrease to zero beyond a certain distance so we need a rapidly decreasing function.

In nuclear physics, the interaction potential between two nucleons may have different forms such as the square well, Yukawa potential, Gaussian potential, or more complicated forms. Actually, the results are independent of the shape assumed for the potential. The common characteristic of these potentials is that they depend only on the inter-nucleon distance [2].

In [10], Abdul-Rahman, Alstaty and Jaqaman used a square well potential which is easier to handle analytically. But in nature, the potential is not expected to be so sharply cut off.

In our work we propose a more realistic form of the potential, Gaussian potential, as will be discussed later in chapter 5

CHAPTER 3

THE WAVEFUNCTION OF THE DEUTERON-NUCLEON SYSTEM

In the previous chapter we discussed the main properties of the deuteron and the main features of the nucleon - nucleon interaction. In this chapter we will use our knowledge about this interaction to construct the wavefunction of the deuteron-nucleon system, which consists of a deuteron and an external nucleon in a low density medium, which interact via Gaussian interaction.

3.1 CONSTRUCTION OF THE DEUTERON-NUCLEON WAVEFUNCTION

The deuteron is a two-body system which can easily be converted to a one-body system after separating the center of mass motion from the internal motion. In light of separation, the deuteron wave function, $\psi_d(\vec{r}_1,\vec{r}_2)$, can be written as:

$$\psi_d(\vec{r}_1,\vec{r}_2) = \psi_{CM}\left(\frac{r1+r2}{2}\right)g(\vec{r}_{12}) \tag{3.1}$$

Where $\psi_{CM}\left(\frac{r1+r2}{2}\right)$ is the center of mass wavefunction, $g(\vec{r}_{12})$ is the internal wavefunction, $\vec{r}_1$ and $\vec{r}_2$ are the positions of the proton and the neutron inside the deuteron. $\vec{r}_{12}$ is the relative distance between the two nucleons with respect to each other ($\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$).

Assuming that the deuteron has a CM momentum $\hbar$K and is confined inside a box of volume L^3, where L is much larger than the size of the deuteron, then the spatial wavefunction of the deuteron is

$$\psi_d(\vec{r}_1,\vec{r}_2) = \frac{1}{L^{3/2}} e^{i\vec{K}.\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)} g(\vec{r}_{12}) \tag{3.2}$$

Now, for the deuteron-nucleon system, assuming without loss of generality, that the external nucleon is a neutron with position $\vec{r}_3$ and momentum $\hbar$k and spatial wavefunction

$$\psi_n(\vec{r}_3) = \frac{1}{L^{3/2}} e^{i\vec{k}.\vec{r}_3} \tag{3.3}$$

Now, the spatial wavefunction of the deuteron-nucleon system will be

$$\emptyset(\vec{r}_1,\vec{r}_2,\vec{r}_3) = g(\vec{r}_{12}) \frac{1}{L^3} e^{i\vec{K}.\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)} e^{i\vec{k}.\vec{r}_3} \tag{3.4}$$

The form of $g(\vec{r}_{12})$ depends on the type of potential used to describe the interaction between the nucleons inside the deuteron. In Abdul-Rahman, Alstaty and Jaqaman [10] they used the square well potential, but in this work we will use the Gaussian potential.

The deuteron-nucleon system has two identical particles; the free neutron and the bound one inside the deuteron. The neutrons are fermions, hence the total wavefunction (spatial and spin parts) of the system should be antisymmetric under the exchange of the two neutrons, so we have to take the spin part to be symmetric if the spatial part is antisymmetric, and vice versa. As a result, the total wavefunction of the system is given by

$$\Psi_{\mathrm{tot}}(\vec{r}_1,\vec{r}_2,\vec{r}_3) = \sqrt{\frac{1}{4}}\{\Psi_{\mathrm{a}}(\vec{r}_1,\vec{r}_2,\vec{r}_3)\sum_{m=-1}^{1}\chi_{1\mathrm{m}}\,(2,3) + \Psi_{\mathrm{s}}(\vec{r}_1,\vec{r}_2,\vec{r}_3)\chi_{00}(2,3)\} \qquad (3.5)$$

Where $\chi_{1\mathrm{m}}(2{,}3)$ and $\chi_{00}(2,3)$ are the 3 triplet and 1 singlet spin states for the two neutrons. We ignore the spin of the third non-identical particle (the proton) as the wavefunction symmetrization only applies to identical particles.

The bound nucleons inside the deuteron are found in the triplet spin state only as we mentioned in the previous chapter, but in equation (3.5) we also take the singlet spin state into consideration because we study the interaction between the nucleons inside the nucleus and the free nucleon, and they can interact through the triplet and singlet interactions with probability $\frac{3}{4}$ and $\frac{1}{4}$ respectively.

In equation (3.5)

$$\Psi_{\mathrm{a}} = \mathrm{N}[\emptyset(\vec{r}_1,\vec{r}_2,\vec{r}_3) - \emptyset(\vec{r}_1\,,\vec{r}_3,\vec{r}_2)] \qquad (3.6)$$

$$\Psi_{s} = \mathrm{N}^{'}[\emptyset(\vec{r}_1,\vec{r}_2,\vec{r}_3) + \emptyset(\vec{r}_1\,,\vec{r}_3,\vec{r}_2)] \qquad (3.7)$$

which are respectively the antisymmetric and the symmetric forms of the spatial wavefunction $\emptyset$ which is defined in equation (3.4).

The normalization constants N and $\mathrm{N}^{'}$ are given by

$$\mathrm{N} = \frac{1}{\sqrt{2}\sqrt{1-\left[{|\mathrm{J}|^2}/{\mathrm{L}^3}\right]}} \quad \text{and} \quad \mathrm{N}' = \frac{1}{\sqrt{2}\sqrt{1+\left[{|\mathrm{J}|^2}/{\mathrm{L}^3}\right]}} \qquad (3.8)$$

Where $$\mathrm{J} = \int g(r)e^{i\left(\frac{\vec{K}}{2}-\vec{k}\right)\cdot\,\vec{r}}\,d^3r \qquad (3.9)$$

It is found typically that $|\mathrm{J}|^2 \ll L^3$, so both N and $\acute{N} \rightarrow \frac{1}{\sqrt{2}}$ [10].

3.2 ENERGY EXPECTATION VALUE FOR THE DEUTERON-NUCLEON SYSTEM

The total wavefunction of the system is found in equation (3.5) to be

$$\Psi_{\text{tot}}(\vec{r}_1,\vec{r}_2,\vec{r}_3) = \sqrt{\frac{1}{4}}\left\{\Psi_{\text{a}}(\vec{r}_1,\vec{r}_2,\vec{r}_3)\sum_{m=-1}^{1}\chi_{1\text{m}}(2,3) + \Psi_{\text{s}}(\vec{r}_1,\vec{r}_2,\vec{r}_3)\chi_{00}(2,3)\right\}$$

Since the triplet and singlet spin wavefunctions are orthogonal then the expectation value of the energy for the system is

$$\langle\Psi_{\text{tot}}|\mathcal{H}|\Psi_{\text{tot}}\rangle = \tfrac{3}{4}\langle\Psi_{\text{a}}|\mathcal{H}|\Psi_{\text{a}}\rangle + \tfrac{1}{4}\langle\Psi_{\text{s}}|\mathcal{H}|\Psi_{\text{s}}\rangle \qquad (3.10)$$

Where $\mathcal{H}$ is the Hamiltonian of the system.

By using the equation (3.6), the first term in the equation (3.10) above will be simplified as follows

$$\langle\Psi_a|\mathcal{H}|\Psi_a\rangle = N^2\,[<\emptyset(1,2,3)|\mathcal{H}(1,2,3)|\emptyset(1,2,3)> - <\emptyset(1,2,3)|\mathcal{H}(1,2,3)|\emptyset(1,3,2)>$$

$$- <\emptyset(1,3,2)|\mathcal{H}(1,3,2)|\emptyset(1,2,3)> + <\emptyset(1,3,2)|\mathcal{H}(1,3,2)|\emptyset(1,3,2)>] \qquad (3.11)$$

The Hamiltonian of the system should be symmetric in the two neutrons 2 and 3,so $\mathcal{H}(1,2,3) = \mathcal{H}(1,3,2) = \mathcal{H}$.

Moreover, we integrate over $\vec{r}_2\ and\ \vec{r}_3$ so we can exchange the labels of 2 and 3 in the last two terms. Then we can see that the first and the last terms are the same and the second and the third terms are the same, so that

$$\langle \Psi_a | \mathcal{H} | \Psi_a \rangle = 2N^2 \, [< \emptyset(1,2,3) | \mathcal{H}(1,2,3) | \emptyset(1,2,3) >$$

$$- < \emptyset(1,2,3) | \mathcal{H}(1,2,3) | \emptyset(1,3,2) >] \qquad (3.12)$$

$\langle \Psi_s | \mathcal{H} | \Psi_s \rangle$ is the same as $\langle \Psi_a | \mathcal{H} | \Psi_a \rangle$ but with a positive sign instead of the negative sign between the two terms in equation (3.12).

The Hamiltonian $\mathcal{H}(1,2,3)$ can be written as

$$\mathcal{H}(1,2,3) = h_{12} - \frac{\hbar^2}{2M} \nabla^2_{R_{12}} - \frac{\hbar^2}{2m} \nabla^2_{r_3} + V(r_{13}) + V(r_{23}) \qquad (3.13)$$

Where h_{12} is the internal Hamiltonian of the deuteron with the reduced mass μ

$$h_{12} = -\frac{\hbar^2}{2\mu} \nabla^2_{r_{12}} + V(r_{12}) \qquad (3.14)$$

the second term in equation (3.13) is the kinetic energy of the CM motion of the deuteron with total mass M=2m, the third term is the kinetic energy of the external neutron with mass m, and V is the nucleon-nucleon interaction. Now, equation (3.12) will be

$$\langle \Psi_a | \mathcal{H} | \Psi_a \rangle = \langle \Psi_a | h_{12} | \Psi_a \rangle + \left\langle \Psi_a \middle| -\frac{\hbar^2}{2M} \nabla^2_{R_{12}} \middle| \Psi_a \right\rangle$$

$$+ \left\langle \Psi_a \middle| -\frac{\hbar^2}{2m} \nabla^2_{r_3} \middle| \Psi_a \right\rangle + \langle \Psi_a | V(r_{13}) + V(r_{23}) | \Psi_a \rangle \qquad (3.15)$$

The first term $\langle \Psi_a | h_{12} | \Psi_a \rangle = -B_0$ is the binding energy of an isolated deuteron, the second term $\left\langle \Psi_a \middle| -\frac{\hbar^2}{2M} \nabla^2_{R_{12}} \middle| \Psi_a \right\rangle = \frac{\hbar^2 K^2}{2M}$ is the kinetic energy of the cluster (deuteron), The third term $\left\langle \Psi_{\mathrm{a}} \middle| -\frac{\hbar^2}{2m}\nabla_{r_3}^2 \middle| \Psi_{\mathrm{a}} \right\rangle = \frac{\hbar^2 k^2}{2m}$ is the kinetic energy of the free nucleon. To find the fourth term we will use equation (3.6)

$$\langle \Psi_{\mathrm{a}} | V(r_{13}) + V(r_{23}) | \Psi_{\mathrm{a}} \rangle = 2N^2 \langle \emptyset(1,2,3) | V(r_{13}) + V(r_{23}) | \emptyset(1,2,3) \rangle$$

$$-2N^2 \langle \emptyset(1,2,3) | V(r_{13}) + V(r_{23}) | \emptyset(1,3,2) \rangle \qquad (3.16)$$

The first term in the equation (3.16):

$$2N^2 \langle \emptyset(1,2,3) | V(r_{13}) + V(r_{23}) | \emptyset(1,2,3) \rangle$$

$$= \frac{2N^2}{L^6} \iiint g^*(r_{12}) e^{-i\vec{K}.\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)} e^{-i\vec{k}.\vec{r}_3} [V(r_{13}) + V(r_{23})]\, g(r_{12}) e^{i\vec{K}.\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)} e^{i\vec{k}.\vec{r}_3}\, d^3r_1 d^3r_2 d^3r_3$$

$$= \frac{2N^2}{L^6} \iiint |g(r_{12})|^2 [V(r_{13}) + V(r_{23})]\, d^3r_1 d^3r_2 d^3r_3$$

$$= \frac{2N^2}{L^6} \iiint |g(r_{12})|^2 V(r_{13})\, d^3r_1 d^3r_2 d^3r_3 + \frac{2N^2}{L^6} \iiint |g(r_{12})|^2 V(r_{23})\, d^3r_1 d^3r_2 d^3r_3 \quad (3.16.a)$$

Now the second term in (3.16):

$$2N^2 \langle \emptyset(1,2,3) | V(r_{13}) + V(r_{23}) | \emptyset(1,3,2) \rangle$$

$$= \frac{2N^2}{L^6} \iiint g^*(r_{12}) e^{-i\vec{K}.\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)} e^{-i\vec{k}.\vec{r}_3} [V(r_{13}) + V(r_{23})] g(r_{13}) e^{i\vec{K}.\left(\frac{\vec{r}_1+\vec{r}_3}{2}\right)} e^{i\vec{k}.\vec{r}_2}\, d^3r_1 d^3r_2 d^3r_3$$

$$= \frac{2N^2}{L^6} \iiint g^*(r_{12}) e^{-i\vec{K}.\left(\frac{\vec{r}_2-\vec{r}_3}{2}\right)} e^{i\vec{k}.(\vec{r}_2-\vec{r}_3)} [V(r_{13}) + V(r_{23})] g(r_{13})\, d^3r_1 d^3r_2 d^3r_3$$

$$= \frac{2N^2}{L^6} \iiint g^*(r_{12}) e^{-i\vec{K}.\left(\frac{\vec{r}_2-\vec{r}_3}{2}\right)} e^{i\vec{k}.(\vec{r}_2-\vec{r}_3)} V(r_{13}) g(r_{13})\, d^3r_1 d^3r_2 d^3r_3$$

$$+ \frac{2N^2}{L^6} \iiint g^*(r_{12}) e^{-i\vec{K}.\left(\frac{\vec{r}_2-\vec{r}_3}{2}\right)} e^{i\vec{k}.(\vec{r}_2-\vec{r}_3)} V(r_{23}) g(r_{13})\, d^3r_1 d^3r_2 d^3r_3 \qquad (3.16.b)$$

$$\langle\Psi_{\mathrm{a}}|\mathcal{H}|\Psi_{\mathrm{a}}\rangle = -B_0 + \frac{\hbar^2K^2}{2M} + \frac{\hbar^2k^2}{2m}$$

$$+\frac{2N^2}{L^6}\iiint |g(r_{12})|^2 V(r_{13})\, d^3r_1 d^3r_2 d^3r_3 + \frac{2N^2}{L^6}\iiint |g(r_{12})|^2 V(r_{23})\, d^3r_1 d^3r_2 d^3r_3$$

$$-\frac{2N^2}{L^6}\iiint g^*(r_{12}) e^{-i\vec{K}.\left(\frac{\vec{r}_2-\vec{r}_3}{2}\right)} e^{i\vec{k}.(\vec{r}_2-\vec{r}_3)} V(r_{13}) g(r_{13})\, d^3r_1 d^3r_2 d^3r_3$$

$$-\frac{2N^2}{L^6}\iiint g^*(r_{12}) e^{-i\vec{K}.\left(\frac{\vec{r}_2-\vec{r}_3}{2}\right)} e^{i\vec{k}.(\vec{r}_2-\vec{r}_3)} V(r_{23}) g(r_{13})\, d^3r_1 d^3r_2 d^3r_3 \qquad (3.17)$$

As we mentioned above, B_0 is the binding energy of the free deuteron, the second and the third terms are the kinetic energies of the deuteron and the external neutron. The fourth and fifth terms represent the self energy, while the sixth and seventh terms represent the Pauli blocking. The kinetic energy terms do not contribute to the binding energy. The self energy is the interaction between the nucleons inside the deuteron and the free nucleon and it is almost the same whether the nucleons are bound to form the deuteron or unbound. So when we calculate the difference between the expectation value of the Hamiltonian when the constituents of the deuteron are bound and when they are unbound

$$\langle\Psi_{\mathrm{tot}}|\mathcal{H}|\Psi_{\mathrm{tot}}\rangle - \langle\Psi_{\mathrm{tot,un}}|\mathcal{H}|\Psi_{\mathrm{tot,un}}\rangle$$

the value of self energy is negligibly small and we can ignore it. We are left with the Pauli blocking terms which are the sixth and the seventh terms, but the seventh term in equation (3.17) is very small and we can neglect it[10], so the most important term is the sixth term which contains $V(r_{13})$.

The nucleons inside the deuteron always interact with each other via the triplet interaction but at the same time they interact with the free nucleon via the triplet interaction with probability of 3/4 and singlet interaction with probability of 1/4. So $V(r_{13})=\frac{3}{4}V_t+\frac{1}{4}V_s$ where the subscripts t and s refer to the triplet and singlet interactions respectively.

So the 6th term in equation (3.17) will be

$$-\frac{2N^2}{L^6}\iiint g^*(r_{12})e^{-i\vec{K}.\left(\frac{\vec{r}_2-\vec{r}_3}{2}\right)}e^{i\vec{k}.(\vec{r}_2-\vec{r}_3)}\left[\frac{3}{4}V_t+\frac{1}{4}V_s\right]g(r_{13})\,d^3r_1d^3r_2d^3r_3$$

$$=-\frac{2N^2}{L^6}\iiint g^*(r_{12})e^{-i\left(\frac{\vec{K}}{2}-\vec{k}\right).(\vec{r}_2-\vec{r}_3)}\left[\frac{3}{4}V_t+\frac{1}{4}V_s\right]g(r_{13})\,d^3r_1d^3r_2d^3r_3$$

To solve the integral we will make the following transformation:

$$\vec{r}_1=\vec{r}_1\,,\vec{r}_{12}=\vec{r}_2-\vec{r}_1\,,\vec{r}_{13}=\vec{r}_3-\vec{r}_1$$

The Jacobian for the transformation above is $\begin{vmatrix}\frac{\partial\vec{r}_1}{\partial\vec{r}_1} & \frac{\partial\vec{r}_1}{\partial\vec{r}_{12}} & \frac{\partial\vec{r}_1}{\partial\vec{r}_{13}}\\ \frac{\partial\vec{r}_2}{\partial\vec{r}_1} & \frac{\partial\vec{r}_2}{\partial\vec{r}_{12}} & \frac{\partial\vec{r}_2}{\partial\vec{r}_{13}}\\ \frac{\partial\vec{r}_3}{\partial\vec{r}_1} & \frac{\partial\vec{r}_3}{\partial\vec{r}_{12}} & \frac{\partial\vec{r}_3}{\partial\vec{r}_{13}}\end{vmatrix}$ which is equal to one.

We will illustrate this by evaluating the Jacobian in one dimension for x :

$x_{11}=x_1$, $x_{12}=x_2-x_1$, $x_{13}=x_3-x_1$

So we can write $x_1, x_2\ and\ x_3$ in terms of $x_{11}, x_{12}\ and\ x_{13}$

$$x_1=x_{11},\qquad x_2=x_{11}+x_{12},\qquad x_3=x_{11}+x_{13}$$

$$\begin{vmatrix} \frac{\partial x_1}{\partial x_{11}} & \frac{\partial x_1}{\partial x_{12}} & \frac{\partial x_1}{\partial x_{13}} \\ \frac{\partial x_2}{\partial x_{11}} & \frac{\partial x_2}{\partial x_{12}} & \frac{\partial x_2}{\partial x_{13}} \\ \frac{\partial\, x_3}{\partial x_{11}} & \frac{\partial\, x_3}{\partial x_{12}} & \frac{\partial\, x_3}{\partial x_{13}} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1$$

Since the Jacobian for the above transformation equals one we can replace $d^3r_1d^3r_2d^3r_3$ by $d^3r_1d^3r_{12}d^3r_{13}$.

And we can write $\vec{r}_2 - \vec{r}_3$ using the new variables $\vec{r}_{12}$and $\vec{r}_{13}$:

$$\vec{r}_2 - \vec{r}_3 = \vec{r}_2 - \vec{r}_1 + \vec{r}_1 - \vec{r}_3 = \vec{r}_{12} - \vec{r}_{13}$$

So the sixth term in the equation (3.17)

$$-\frac{2N^2}{L^6}\iiint g^*(r_{12})e^{-i\left(\frac{\vec{K}}{2}-\vec{k}\right).(\vec{r}_{12}-\vec{r}_{13})}\left[\frac{3}{4}V_t(r_{13})+\frac{1}{4}V_s(r_{13})\right]g(r_{13})\,d^3r_1d^3r_{12}d^3r_{13}$$

can be written as:

$$-\frac{2N^2}{L^6}\int d^3r_1\int g^*(r_{12})e^{-i\left(\frac{\vec{K}}{2}-\vec{k}\right).\vec{r}_{12}}d^3r_{12}\int g(r_{13})\left[\frac{3}{4}V_t+\frac{1}{4}V_s\right]e^{i\left(\frac{\vec{K}}{2}-\vec{k}\right).\vec{r}_{13}}d^3r_{13}$$

$$= -\frac{2N^2}{L^6}L^3\mathrm{J}^*.\left[\frac{3}{4}\mathrm{J}_{2\mathrm{t}}+\frac{1}{4}\mathrm{J}_{2\mathrm{s}}\right]$$

Where $\int d^3r_1 = L^3$ which is the volume of the box where the system is confined. And from equation (3.9)

$$\mathrm{J} = \int g(r_{12})e^{i\left(\frac{\vec{K}}{2}-\vec{k}\right).\vec{r}_{12}}\,d^3r_{12}$$

so the complex conjugate $\mathrm{J}^* = \int g^*(r_{12})e^{-i\left(\frac{\vec{K}}{2}-\vec{k}\right).\vec{r}_{12}}\,d^3r_{12}$

$$\mathrm{J}_{2\mathrm{t}} = \int g(r_{13}) V_t(r_{13}) e^{i\left(\frac{\vec{K}}{2}-\vec{k}\right).\vec{r}_{13}} d^3 r_{13} \tag{3.18}$$

$$\mathrm{J}_{2\mathrm{s}} = \int g(r_{13}) V_s(r_{13}) e^{i\left(\frac{\vec{K}}{2}-\vec{k}\right).\vec{r}_{13}} d^3 r_{13} \tag{3.19}$$

Therefore

$$\langle \Psi_{\mathrm{a}} | \mathcal{H} | \Psi_{\mathrm{a}} \rangle = -B_0 - \frac{2N^2}{L^3} \mathrm{J}^*. \left[\frac{3}{4}\mathrm{J}_{2\mathrm{t}} + \frac{1}{4}\mathrm{J}_{2\mathrm{s}}\right] \tag{3.20a}$$

$$\langle \Psi_{\mathrm{s}} | \mathcal{H} | \Psi_{\mathrm{s}} \rangle = -B_0 + \frac{2N^{'2}}{L^3} \mathrm{J}^*. \left[\frac{3}{4}\mathrm{J}_{2\mathrm{t}} + \frac{1}{4}\mathrm{J}_{2\mathrm{s}}\right] \tag{3.20b}$$

$$\langle \Psi_{\mathrm{tot}} | \mathcal{H} | \Psi_{\mathrm{tot}} \rangle = \frac{3}{4} \langle \Psi_{\mathrm{a}} | \mathcal{H} | \Psi_{\mathrm{a}} \rangle + \frac{1}{4} \langle \Psi_{\mathrm{s}} | \mathcal{H} | \Psi_{\mathrm{s}} \rangle$$

$$= -B_0 - \frac{3}{4}\frac{2N^2}{L^3} \mathrm{J}^*. \left[\frac{3}{4}\mathrm{J}_{2\mathrm{t}} + \frac{1}{4}\mathrm{J}_{2\mathrm{s}}\right] + \frac{1}{4}\frac{2N^{'2}}{L^3} \mathrm{J}^*. \left[\frac{3}{4}\mathrm{J}_{2\mathrm{t}} + \frac{1}{4}\mathrm{J}_{2\mathrm{s}}\right] \tag{3.21}$$

Noting that $N$and $\acute{N}\ \ \rightarrow \frac{1}{\sqrt{2}}$. Equation.(3.21) reduces to:

$$\langle \Psi_{\mathrm{tot}} | \mathcal{H} | \Psi_{\mathrm{tot}} \rangle = -B_0 - \frac{3}{8}\frac{1}{L^3} \mathrm{J}^*. \mathrm{J}_{2\mathrm{t}} - \frac{1}{8}\frac{1}{L^3}. \mathrm{J}^*. \mathrm{J}_{2\mathrm{s}} \tag{3.22}$$

3.3 BINDING ENERGY FORMULA FOR A DEUTERON IMMERSED IN A VAPOR OF NUCLEONS.

All the above calculations were done for one external neutron, to include the contribution of other nucleons in the vapor we assume that there are n nucleons in the volume L^3, so we can use the number density $\rho = \frac{n}{L^3}$ to include all nucleons in the vapor, so the energy in equation (3.22) can be written as:

$$E = \langle \Psi_{\mathrm{tot}} | \mathcal{H} | \Psi_{\mathrm{tot}} \rangle = -B_0 - \frac{3}{8}\rho . \mathrm{J}^* . \mathrm{J}_{2\mathrm{t}} - \frac{1}{8}\rho . \mathrm{J}^* . \mathrm{J}_{2\mathrm{s}} \tag{3.23}$$

and hence the binding energy formula for a deuteron immersed in a nucleon vapor by considering the Pauli blocking shift becomes:

$$B(\rho) = -\langle \Psi_{\mathrm{tot}} | \mathcal{H} | \Psi_{\mathrm{tot}} \rangle = B_0 + \frac{3}{8}\rho . \mathrm{J}^* . \mathrm{J}_{2\mathrm{t}} + \frac{1}{8}\rho . \mathrm{J}^* . \mathrm{J}_{2\mathrm{s}} \tag{3.24}$$

The integrals $\mathrm{J}, \mathrm{J}_{2\mathrm{t}}$ and $\mathrm{J}_{2\mathrm{s}}$ depend on the wavevectors $\vec{K}$and $\vec{k}$ for the deuteron and the free nucleon, and these quantities depend on the system temperature. Therefore, we will find the statistical average over all momentum space, we will use the Nuclear Statistical Equilibrium (NSE) and the chemical potentials for ideal Bose and Fermi gases [24]. All of these calculations will be done in the next chapter.

CHAPTER 4

THE CENTER OF MASS MOMENTA OF DEUTERONS AND NUCLEONS

In the previous chapter we derived the formula for binding energy for a deuteron immersed in a vapor of nucleons by taking into consideration the Pauli blocking effect.

$$B(\rho) = B_0 + \frac{3}{8}\rho . \mathrm{J}^* . \mathrm{J}_{2\mathrm{t}} + \frac{1}{8}\rho . \mathrm{J}^* . \mathrm{J}_{2\mathrm{s}} \tag{4.1}$$

The integrals $\mathrm{J}, \mathrm{J}_{2\mathrm{t}}$ and $\mathrm{J}_{2\mathrm{s}}$ were defined in equations (3.9), (3.18), and (3.19). All of these equations depend on the quantity $\frac{\vec{K}}{2} - \vec{k}$, where $\vec{K}$and $\vec{k}$ are the wavevectors for the deuteron and the free nucleon respectively.

In this chapter we will evaluate the integrals $\mathrm{J}, \mathrm{J}_{2\mathrm{t}}$ and $\mathrm{J}_{2\mathrm{s}}$

$$\mathrm{J} = \int g(r) e^{i\vec{Q}.\vec{r}}\, d^3r \tag{4.2}$$

$$\mathrm{J}_{2\mathrm{t}} = \int g(r) V_t(r) e^{i\overrightarrow{Q.\vec{r}}} d^3r \tag{4.3}$$

$$\mathrm{J}_{2\mathrm{s}} = \int g(r) V_s(r) e^{i\vec{Q}.\vec{r}} d^3r \tag{4.4}$$

Where $\vec{Q} = \frac{\vec{K}}{2} - \vec{k}$ depends on the system's temperature. Therefore, to evaluate the integrals we will find the statistical average for the exponential functions that include Q; $\langle e^{i\vec{Q}.\vec{r}} \rangle$. In our system we have a nucleon (which is a fermion) with momentum $\hbar\vec{k}$ and a deuteron (which is a boson) with momentum $\hbar\vec{K}$. So we will use the statistical mechanics of Fermi and Bose gases.

To find the average values of physical quantities for a system composed of fermions, we should use the Fermi-Dirac distribution function [25]. Suppose that we have a system composed of $\mathcal{A}$ fermions with single particle energies labeled as $\varepsilon_1, \varepsilon_2, \dots \varepsilon_A$, the Fermi-Dirac distribution function is given by

$$f_{FD} = \frac{1}{e^{\beta(\varepsilon_i - \mu)} + 1} \tag{4.5}$$

Where μ is the chemical potential, and $\beta = \frac{1}{k_B T}$. k_B is Boltzmann constant and T is the temperature. From equation (4.5) we can notice that the Fermi-Dirac distribution function cannot be more than 1 or less than 0. At low temperatures (very close to absolute zero T = 0) the behavior of the function depends on the value of $(\varepsilon_i - \mu)$. If $(\varepsilon_i - \mu) < 0$ and $|\varepsilon_i - \mu| \gg k_B T$, the Fermi Dirac distribution function tends to its maximum 1. But if $(\varepsilon_i - \mu) > 0$and $|\varepsilon_i - \mu| \gg k_B T$, the function tends to its minimum 0 [25].

At these low temperatures (T ≈ 0), fermions are not like bosons they cannot share the same state because of the Pauli exclusion principle. But they can occupy the lowest distinct energy states up to the Fermi energy ε_F which is the energy of the highest possible occupied state [23].

In this case, the Fermi-Dirac distribution function is defined as

$$f_{FD} = \begin{cases} 1, & \varepsilon \leq \varepsilon_F \\ 0, & \varepsilon > \varepsilon_F \end{cases}$$

To find the average values of physical quantities for systems composed of bosons, we have to use the Bose-Einstein distribution function [25]. For a system composed of$\mathcal{A}'$ bosons with single particle energies labeled as $\acute{\varepsilon}_1, \acute{\varepsilon}_2, \ldots \acute{\varepsilon}_{\acute{A}}$ and chemical potential $\acute{\mu}$, the Bose-Einstein distribution function is given by

$$f_{BE} = \frac{1}{e^{\beta(\acute{\varepsilon}_i - \acute{\mu})} - 1} \tag{4.6}$$

At low temperatures (very close to absolute zero T = 0), a large fraction of bosons accumulate in the ground state which leads to the phenomenon of Bose-Einstein condensation. Such phenomenon cannot happen in a Fermi gas because of the Pauli exclusion principle.

As we see in the equations (4.5) and (4.6) for fermions and bosons, they look the same, but they only differ in the negative sign in the denominator. So the derivation of $\langle e^{-i\vec{k}.\vec{r}} \rangle$ is similar to derivation of$\langle e^{i\vec{K}.\vec{r}/2} \rangle$. Now we can start deriving $\langle e^{-i\vec{k}.\vec{r}} \rangle$ using the Fermi-Dirac distribution function.

4.1 EVALUATION OF $\langle e^{i\vec{K}.\vec{r}/2} \rangle$AND$\langle e^{-i\vec{k}.\vec{r}} \rangle$AT HIGH TEMPERATURES

As we mentioned above we can derive $\langle e^{-i\vec{k}.\vec{r}} \rangle$ using the Fermi-Dirac distribution function. In this model, the gas consists of non-interacting indistinguishable fermions. At absolute zero (T=0), the Fermi-Dirac distribution function has a special behavior; it is unity for all states with $\varepsilon < \mu$ and zero for all states with $\varepsilon > \mu$. We will discuss this case in section 4.3.

At any higher temperature, some particles will occupy higher energy states above the fermi energy. In our work we consider an ideal Fermi system of n non-interacting particles in a cubical box of volume L^3 with a wavefunction of the form

$$\psi_n(\vec{r}) = \frac{1}{L^{3/2}} e^{i\vec{k}.\vec{r}} \tag{4.7}$$

The $\vec{k}$ is the wave vector of the particle and $\vec{r}$ is the position vector. The wave vector components can be written as:

$$k_x = \frac{2\pi}{L} n_x, \qquad k_y = \frac{2\pi}{L} n_y, \qquad k_z = \frac{2\pi}{L} n_z \tag{4.8}$$

Where n_x, n_y and n_z are $0, \pm 1, \pm 2, \ldots$.

The number of allowed plane wave states in a volume element d^3k is

$$dn = 4\left(\frac{L}{2\pi}\right)^3 d^3k \tag{4.9}$$

Where the number 4 is the spin-isospin degeneracy factor for nucleons. It is a weight factor that arises from the internal structure of particles such as spin. For nucleons,

2 x 2 = 4 the factors of 2 come from the two spin states and the two isospin states of the nucleons, which are the proton and the neutron. Therefore, the total number of nucleons ($\mathcal{A}$) is given by

$$\mathcal{A} = 4\left(\frac{L}{2\pi}\right)^3 \int f_{FD}\, d^3k \tag{4.10}$$

and the average value $\langle e^{-i\vec{k}.\vec{r}} \rangle = \frac{4}{\mathcal{A}}\left(\frac{L}{2\pi}\right)^3 \int e^{-i\vec{k}.\vec{r}} f_{FD}\, d^3k$ (4.11)

To find $\langle e^{-i\vec{k}.\vec{r}}\rangle$ at high temperatures we will use the method adopted by Jaqaman *et al*[6]to expand the Fermi-Dirac distribution function. Using the same method we can find the chemical potential μ at high temperatures.

At high temperatures $[(\varepsilon_i - \mu) \ll k_B T]$, the Fermi system is said to be partially degenerate and hence the occupation probability for the state ε_i is much smaller than unity $\left[\frac{(\varepsilon_i - \mu)}{k_B T} \ll 1\right]$.

The expansion below is suitable for such cases.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n, \text{where} |x| < 1 \tag{4.12}$$

We can find

$$f_{FD} = \frac{1}{e^{\beta(\varepsilon_i-\mu)}+1} = \frac{e^{-\beta(\varepsilon_i-\mu)}}{1+e^{-\beta(\varepsilon_i-\mu)}}$$

$$= e^{-\beta(\varepsilon_i-\mu)}\left[1 - e^{-\beta(\varepsilon_i-\mu)} + e^{-2\beta(\varepsilon_i-\mu)} - \cdots\right]$$

$$= e^{-\beta(\varepsilon_i-\mu)} - e^{-2\beta(\varepsilon_i-\mu)} + e^{-3\beta(\varepsilon_i-\mu)} + \cdots + (-1)^{n+1}e^{-n\beta(\varepsilon_i-\mu)} + \cdots$$

$$= f(T) - f\left(\frac{T}{2}\right) + f\left(\frac{T}{3}\right) - f\left(\frac{T}{4}\right) + \cdots + (-1)^{n+1} f\left(\frac{T}{n}\right) + \cdots \tag{4.13}$$

Where
$$f\left(\frac{T}{n}\right) = e^{-n\beta(\varepsilon_i-\mu)} \tag{4.14}$$

The temperature T is implicitly included in $\beta = \frac{1}{k_B T}$

Substituting equation (4.13) in equation (4.11), we get

$$\langle e^{-i\vec{k}.\vec{r}} \rangle = \frac{4}{\mathcal{A}}\left(\frac{L}{2\pi}\right)^3 [\int e^{-i\vec{k}.\vec{r}} f(T)\, d^3k - \int e^{-i\vec{k}.\vec{r}} f\left(\frac{T}{2}\right) d^3k + \cdots$$

$$+(-1)^{n+1} \int e^{-i\vec{k}.\vec{r}} f\left(\frac{T}{n}\right) d^3k + \cdots] \qquad (4.15)$$

$$= \frac{4}{\mathcal{A}}\left(\frac{L}{2\pi}\right)^3 [I_1 - I_2 + \cdots + (-1)^{n+1} I_n + \cdots]$$

Where I_n is given by

$$I_n = \int e^{-i\vec{k}.\vec{r}} f\left(\frac{T}{n}\right) d^3k$$

$$= e^{n\mu/k_BT} \int e^{-i\vec{k}.\vec{r}} e^{-n\hbar^2k^2/2mk_BT}\, d^3k \qquad (4.16)$$

Now, we will evaluate the integrals between the brackets in equation (4.15) by finding a general formula for the n^{th} integral I_n instead of evaluating them one by one.

We assumed there is no interaction between the nucleons so the single particle energy $\varepsilon_i = \frac{\hbar^2k^2}{2m}$ is purely kinetic. To find the integral in equation (4.16), we will use completing the square technique

$$\frac{-n\hbar^2k^2}{2mk_BT} - i\vec{k}.\vec{r} = \frac{-n\hbar^2}{2mk_BT}\left[\vec{k} + \frac{imk_BT\vec{r}}{n\hbar^2}\right]^2 - \frac{mk_BTr^2}{2n\hbar^2} \qquad (4.17)$$

After collecting the exponents, we can substitute equation(4.17) in equation (4.16) to get

$$I_n = e^{n\mu/k_BT} e^{-mk_BTr^2/2n\hbar^2} \int e^{-\frac{-n\hbar^2}{2mk_BT}\left[\vec{k}+\frac{imk_BT\vec{r}}{n\hbar^2}\right]^2}\, d^3k \qquad (4.18)$$

To simplify the integral in equation (4.18), let us set

$$\vec{u} = \vec{k} + \frac{imk_BT\vec{r}}{n\hbar^2} \Longrightarrow \begin{cases} u_x = k_x + \frac{imk_BTx}{n\hbar^2} \\ u_y = k_y + \frac{imk_BTy}{n\hbar^2} \\ u_z = k_z + \frac{imk_BTz}{n\hbar^2} \end{cases} \tag{4.19}$$

hence
$$\left.\begin{matrix} du_x = dk_x \\ du_y = dk_y \\ du_z = dk_z \end{matrix}\right\} \Longrightarrow d^3k = d^3u \tag{4.20}$$

Using equation (4.19) and (4.20), the equation (4.18) becomes

$$I_n = e^{n\mu/k_BT} e^{-mk_BTr^2/2n\hbar^2} \int e^{-\frac{-n\hbar^2u^2}{2mk_BT}} \, d^3u$$

$$= e^{n\mu/k_BT} e^{-mk_BTr^2/2n\hbar^2} (4\pi) \int_0^\infty u^2 e^{-\frac{-n\hbar^2u^2}{2mk_BT}} du$$

We can solve the integral using the following formula:

$$\int_0^\infty v^2 e^{-cv^2} dv = \frac{\sqrt{\pi}}{4c^{3/2}} \tag{4.21}$$

Where c is a constant

So equation (4.18) will be

$$I_n = e^{n\mu/k_BT} e^{-mk_BTr^2/2n\hbar^2} (4\pi) \frac{\sqrt{\pi}}{4\left(\frac{n\hbar^2}{2mk_BT}\right)^{3/2}} \tag{4.22}$$

Now, let us evaluate $\mathcal{A}$ by substituting equation (4.13) in equation (4.10)

$$\mathcal{A} = 4\left(\frac{L}{2\pi}\right)^3 [\int f(T)\, d^3k - \int f\left(\frac{T}{2}\right) d^3k + \int f\left(\frac{T}{3}\right) d^3k - \cdots$$

$$+(-1)^{n+1} \int f\left(\frac{T}{n}\right) d^3k + \cdots]. \tag{4.23}$$

$$= 4\left(\frac{L}{2\pi}\right)^3\left[\acute{I}_1 - \acute{I}_2 + \cdots + (-1)^{n+1}\acute{I}_n + \cdots\right]$$

$f\left(\frac{T}{n}\right)$ is defined in equation (4.14).

The general formula for the nth integral I'_n is

$$I'_n = \int f\left(\frac{T}{n}\right) d^3k = \int e^{-n(\varepsilon_i-\mu)/k_BT}\, d^3k$$

$$= e^{n\mu/k_BT}(4\pi)\int_0^\infty k^2 e^{-n\hbar^2k^2/2mk_BT}\, dk$$

$$= e^{n\mu/k_BT}(4\pi)\frac{\sqrt{\pi}}{4\left(\frac{n\hbar^2}{2mk_BT}\right)^{3/2}} \tag{4.24}$$

Where we used the formula of equation (4.21) again. As a result, using equations (4.10), (4.11), (4.15), (4.23) and (4.24), $\langle e^{-i\vec{k}.\vec{r}}\rangle$ is given by

$$\langle e^{-i\vec{k}.\vec{r}}\rangle = \frac{4\left(\frac{L}{2\pi}\right)^3[I_1-I_2+I_3-I_4+I_5-I_6+I_7-\cdots]}{4\left(\frac{L}{2\pi}\right)^3[I'_1-I'_2+I'_3-I'_4+I'_5-I'_6+I'_7-\cdots]}$$

$$\langle e^{-i\vec{k}.\vec{r}}\rangle = \frac{[I_1-I_2+I_3-I_4+I_5-I_6+I_7-\cdots]}{[I'_1-I'_2+I'_3-I'_4+I'_5-I'_6+I'_7-\cdots]} \tag{4.25}$$

It is sufficient to take the first seven terms $(n = 7)$ in the above equation as it will be shown later in chapter 5.

Now, we can use the same method to derive $\langle e^{i\frac{\vec{K}}{2}.\vec{r}}\rangle$ which is defined as

$$\langle e^{i\frac{\vec{K}}{2}.\vec{r}}\rangle = \frac{3}{\mathcal{A}'}\left(\frac{L}{2\pi}\right)^3 \int e^{i\frac{\vec{K}}{2}.\vec{r}} f_{BE}\, d^3K \tag{4.26}$$

Where $\mathcal{A}'$ is the total number of bosons (deuterons) in the boson gas and is given by

$$\mathcal{A}' = 3\left(\frac{L}{2\pi}\right)^3 \int f_{BE}\, d^3K \qquad (4.27)$$

Where the number 3 here is the spin degeneracy factor for the deuteron. Using the expansion of equation (4.12), we can make the high temperature expansion for the Bose-Einstein distribution function with the chemical potential $\mu'$and single particle energy $\acute{\varepsilon}_i = \frac{\hbar^2 k^2}{2(2m)} = \frac{\hbar^2 k^2}{4m}$

$$f_{BE} = \frac{1}{e^{\beta(\acute{\varepsilon}_i - \mu')} - 1} = \frac{e^{-\beta(\acute{\varepsilon}_i - \mu')}}{1 - e^{-\beta(\acute{\varepsilon}_i - \mu')}} = e^{-\beta(\acute{\varepsilon}_i - \mu')}\left[1 + e^{-\beta(\acute{\varepsilon}_i - \mu')} + \cdots\right]$$

$$= e^{-\beta(\acute{\varepsilon}_i - \mu')} + e^{-2\beta(\acute{\varepsilon}_i - \mu')} + e^{-3\beta(\acute{\varepsilon}_i - \mu')} + \cdots$$

$$= \acute{f}(T) + \acute{f}\left(\frac{T}{2}\right) + \acute{f}\left(\frac{T}{3}\right) + \cdots + \acute{f}\left(\frac{T}{n}\right) + \cdots \qquad (4.28)$$

Where $\acute{f}\left(\frac{T}{n}\right) = e^{-n\beta(\acute{\varepsilon}_i - \mu')}$

In a similar way to the derivation of $\langle e^{-i\vec{k}.\vec{r}}\rangle$, substituting the expansion of equation (4.28) in equation (4.26)

$$\langle e^{i\frac{\vec{K}}{2}.\vec{r}}\rangle = \frac{3}{\mathcal{A}'}\left(\frac{L}{2\pi}\right)^3 \int e^{i\frac{\vec{K}}{2}.\vec{r}}\left[\acute{f}(T) + \acute{f}\left(\frac{T}{2}\right) + \acute{f}\left(\frac{T}{3}\right) + \cdots\right] d^3K$$

$$= \frac{3}{\mathcal{A}'}\left(\frac{L}{2\pi}\right)^3 \left[\int e^{i\frac{\vec{K}}{2}.\vec{r}}\acute{f}(T)\, d^3K + \int e^{i\frac{\vec{K}}{2}.\vec{r}}\acute{f}\left(\frac{T}{2}\right) d^3K + \int e^{i\frac{\vec{K}}{2}.\vec{r}}\acute{f}\left(\frac{T}{3}\right) d^3K + \cdots\right]$$

$$= \frac{3}{\mathcal{A}'}\left(\frac{L}{2\pi}\right)^3 [y_1 + y_2 + y_3 + \cdots + y_n + \cdots.]$$

where $y_n = \int e^{i\frac{\vec{K}}{2}.\vec{r}} \acute{f}\left(\frac{T}{n}\right) d^3K$

$$= \int e^{i\frac{\vec{K}}{2}.\vec{r}}\left[e^{-n\beta(\acute{\varepsilon}_i-\mu')}\right] d^3K = \int e^{i\frac{\vec{K}}{2}.\vec{r}}\left[e^{-n/k_BT(\acute{\varepsilon}_i-\mu')}\right] d^3K$$

By substituting the single particle energy $\acute{\varepsilon}_i = \frac{\hbar^2k^2}{4m}$ and using completing the square technique, the integral y_n will be

$$y_n = e^{n\mu'/k_BT} e^{-mk_BTr^2/4n\hbar^2}(4\pi)\int_0^\infty u^2 e^{-\frac{-n\hbar^2u^2}{4mk_BT}}du$$

$$= e^{n\mu'/k_BT} e^{-mk_BTr^2/4n\hbar^2}(4\pi)\frac{\sqrt{\pi}}{4\left(\frac{n\hbar^2}{4mk_BT}\right)^{3/2}} \tag{4.29}$$

Now let us evaluate $\mathcal{A}'$ by substituting equation (4.28) in equation (4.27)

$$\mathcal{A}' = 3\left(\frac{L}{2\pi}\right)^3 \int f_{BE}\, d^3K = \mathcal{A}' = 3\left(\frac{L}{2\pi}\right)^3 \int \acute{f}(T) + \acute{f}\left(\frac{T}{2}\right) + \acute{f}\left(\frac{T}{3}\right) + \cdots d^3K$$

$$= 3\left(\frac{L}{2\pi}\right)^3 \left[\int\left[\acute{f}(T)\right] d^3K + \int\left[\acute{f}\left(\frac{T}{2}\right)\right] d^3K + \int\left[\acute{f}\left(\frac{T}{3}\right)\right] d^3K + \cdots\right]$$

$$= \frac{3}{\mathcal{A}'}\left(\frac{L}{2\pi}\right)^3 [y'_1 + y'_2 + y'_3 + \cdots + y'_n + \cdots.]$$

where

$$y'_n = \int\left[\acute{f}\left(\frac{T}{n}\right)\right] d^3K = \int\left[e^{-n\beta(\acute{\varepsilon}_i-\mu')}\right] d^3K$$

Now we will evaluate the general formula for the n^{th}integral y'_n.

By substituting the single particle energy $\acute{\varepsilon}_i = \frac{\hbar^2k^2}{4m}$ and using completing the square technique, the integral y'_n will be

$$y_n' = e^{n\mu'/k_BT}(4\pi)\int_0^\infty K^2 e^{-\frac{-n\hbar^2K^2}{4mk_BT}}\,dK$$

$$= e^{n\mu'/k_BT}(4\pi)\frac{\sqrt{\pi}}{4\left(\frac{n\hbar^2}{4mk_BT}\right)^{3/2}} \tag{4.30}$$

As a result, using equations (4.26), (4.27), (4.29) and (4.30), $\langle e^{i\vec{K}.\vec{r}/2}\rangle$ is given by

$$\langle e^{i\vec{K}.\vec{r}/2}\rangle = \frac{3\left(\frac{L}{2\pi}\right)^3[y_1+y_2+y_3+y_4+y_5+y_6+y_7+\cdots]}{3\left(\frac{L}{2\pi}\right)^3[y_1'+y_2'+y_3'+y_4'+y_5'+y_6'+y_7'+\cdots]}$$

$$= \frac{[y_1+y_2+y_3+y_4+y_5+y_6+y_7+\cdots]}{[y_1'+y_2'+y_3'+y_4'+y_5'+y_6'+y_7'+\cdots]} \tag{4.31}$$

Also, it is sufficient to take the first seven terms only ($n = 7$) as we will show in chapter 5.

By substituting equations (4.22) and (4.24) in equation (4.25) and taking the first seven terms, we get the final form of $\langle e^{-i\vec{k}.\vec{r}}\rangle$

$$\langle e^{-i\vec{k}.\vec{r}}\rangle =$$

$$\frac{\dfrac{\pi^{3/2}e^{\mu/k_BT}\left[e^{-mk_BTr^2/2\hbar^2}-\frac{1}{2^{3/2}}e^{\mu/k_BT}e^{-mk_BTr^2/4\hbar^2}+\frac{1}{3^{3/2}}e^{2\mu/k_BT}e^{-mk_BTr^2/6\hbar^2}-\frac{1}{4^{3/2}}e^{3\mu/k_BT}e^{-mk_BTr^2/8\hbar^2}+\cdots\right]}{\left(\frac{\hbar^2}{4mk_BT}\right)^{3/2}}}{\dfrac{\pi^{\frac{3}{2}}e^{\mu/k_BT}\left[1-\frac{1}{2^{3/2}}e^{\mu/k_BT}+\frac{1}{3^{3/2}}e^{2\mu/k_BT}e^{-mk_BTr^2/6\hbar^2}-\frac{1}{4^{3/2}}e^{3\mu/k_BT}e^{-mk_BTr^2/8\hbar^2}+\cdots\right]}{\left(\frac{\hbar^2}{4mk_BT}\right)^{3/2}}}$$

$$=$$

$$\frac{\left[e^{-mk_BTr^2/2\hbar^2}-\frac{1}{2^{3/2}}e^{\mu/k_BT}e^{-mk_BTr^2/4\hbar^2}+\frac{1}{3^{3/2}}e^{2\mu/k_BT}e^{-mk_BTr^2/6\hbar^2}-\frac{1}{4^{3/2}}e^{3\mu/k_BT}e^{-mk_BTr^2/8\hbar^2}+\cdots\right]}{\left[e^{-mk_BTr^2/2\hbar^2}-\frac{1}{2^{3/2}}e^{\mu/k_BT}e^{-mk_BTr^2/4\hbar^2}+\frac{1}{3^{3/2}}e^{2\mu/k_BT}e^{-mk_BTr^2/6\hbar^2}-\frac{1}{4^{3/2}}e^{3\mu/k_BT}e^{-mk_BTr^2/8\hbar^2}+\cdots\right]}$$

$$=\frac{\left[e^{-mk_BTr^2/2\hbar^2}-\frac{1}{2^{3/2}}e^{\mu/k_BT}e^{-mk_BTr^2/4\hbar^2}+\frac{1}{3^{3/2}}e^{2\mu/k_BT}e^{-mk_BTr^2/6\hbar^2}-\frac{1}{4^{3/2}}e^{3\mu/k_BT}e^{-mk_BTr^2/8\hbar^2}+\cdots\right]}{\left[1-\frac{1}{2^{3/2}}e^{\mu/k_BT}+\frac{1}{3^{3/2}}e^{2\mu/k_BT}-\frac{1}{4^{3/2}}e^{3\mu/k_BT}+\cdots\right]}$$

By taking the first seven terms, we get

$$\langle e^{-i\vec{k}.\vec{r}}\rangle=\frac{\sum_{n=1}^{7}\left[\frac{(-1)^{n+1}}{n^{3/2}}e^{\mu(n-1)/k_BT}e^{-mk_BTr^2/2n\hbar^2}\right]}{\sum_{n=1}^{7}\left[\frac{(-1)^{n+1}}{n^{3/2}}e^{\mu(n-1)/k_BT}\right]} \tag{4.32}$$

Following the same steps, by substituting equations (4.29) and (4.30) in (4.31) we get

$$\langle e^{i\vec{K}.\vec{r}/2}\rangle=\frac{\sum_{n=1}^{7}\left[\frac{1}{n^{3/2}}e^{\mu'(n-1)/k_BT}e^{-mk_BTr^2/4n\hbar^2}\right]}{\sum_{n=1}^{7}\left[\frac{1}{n^{3/2}}e^{\mu'(n-1)/k_BT}\right]} \tag{4.33}$$

Now, we can calculate the integrals in equations (4.2), (4.3) and (4.4) using the equations (4.32) and (4.33) for $\langle e^{-i\vec{k}.\vec{r}}\rangle$ and $\langle e^{i\vec{K}.\vec{r}/2}\rangle$. But first we will evaluate the chemical potentials μ and μ'.

4.2 Chemical Potentials for the Deuterons and Nucleons.

In our work we assume thermal and chemical equilibrium and we ignore the interaction between nucleons except for the formation of clusters. Moreover, we viewed the nuclear matter at low density. The Nuclear Statistical Equilibrium (NSE) model suits these conditions [26-28] to relate the chemical potential of the nucleons to the chemical potential of the deuteron.

Due to statistical equilibrium there is chemical equilibrium between the clusters and the nucleons in the vapor [24] such that

$$\mu_C = Z\mu_p + N\mu_n \tag{4.34}$$

Where μ_C, μ_p and μ_n are the chemical potentials for the clusters, protons, and neutrons respectively. Z and N are the numbers of protons and neutrons in the cluster.

Because of the assumed symmetry of nuclear matter, the chemical potentials of the protons and neutrons are equal $\mu_p = \mu_n = \mu$, so that:

$$\mu_{cluster} = A\,\mu \tag{4.35}$$

Where A=N+Z.

For the deuteron A = 2, so

$$\mu' = \mu_{deuteron} = 2\,\mu \tag{4.36}$$

Now, the number density for the deuteron ρ_d is given by

$$\rho_d = \frac{\mathcal{A}'}{L^3} = \frac{g}{(2\pi)^3}\int \acute{f}_d d^3K \tag{4.37}$$

where $\mathcal{A}'$ is the total number of deuterons in the system, $g = 2S + 1 = 3$ is the spin degeneracy factor of the deuteron, and $\acute{f}_d$ is the probability of finding the deuteron cluster with kinetic energy $\acute{\varepsilon} = \frac{\hbar^2 k^2}{4m}$ [24]

$$\acute{f}_d = \frac{1}{e^{\beta(\acute{\varepsilon} - \mu' - B(\rho))} - 1} \tag{4.38}$$

$B(\rho)$ is the density dependent binding energy of the deuteron when embedded in the nucleon vapor.

So all equations with derivations depend on f_{BE}should be modified to include $B(\rho)$.

In all equations include μ' we can replace μ' by $\mu' + B(\rho)$, and this is clear if we compare equation (4.6) with equation (4.38). But μ' for the deuteron equals 2μ

$$\mu' \longrightarrow 2\mu + B(\rho) \tag{4.39}$$

In the equation of binding energy $B(\rho)$, we found that it is a function of the total density ρ. But we can also notice that the values of $\langle e^{i\vec{K}.\vec{r}/2}\rangle$ and $\langle e^{-i\vec{k}.\vec{r}}\rangle$ depend on the binding energy. Therefore, in order to achieve self-consistency many iterative operations should be performed when calculating the binding energy, and the value of the total number density of the system that achieves self-consistency is then used to get the binding energy of the deuteron immersed in nuclear matter. The iteration stops when the difference between two successive values of $B(\rho)$ is less than 0.001 MeV.

Finally, we will find the formula for the chemical potential μ of the free nucleons. The chemical potential of free nucleons is calculated using the equation of state for an infinite system of non-interacting nucleons at low density [24]:

$$\mu(T,\rho) = k_B T \left(ln\left(\frac{\lambda_T^3 \rho}{g}\right) + \sum_{l=1}^{\infty} b_n \left(\frac{\lambda_T^3 \rho}{g}\right)^l \right) \tag{4.40}$$

where $\lambda_T = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{1/2}$ is the thermal wavelength of the nucleon in the gas, and it is defined as the mean de Broglie wavelength of the nucleons in an ideal gas evaluated at temperature T.

The $b_n's$ are the expansion coefficients that were obtained by using the method of series inversion. These coefficients were evaluated in [24]. In our work we will stop at $n = 6$. From the table (4.1) below we can notice that the values of $b_n$rapidly decrease as the index n increases and they are alternative.

In [24] it was shown that at low temperatures (T< 4 MeV) the contribution after the sixth term is negligible at low densities, for example at T = 3 MeV the equation (4.40) is convergent at densities up to 0.12 $nucleons/\mathrm{fm}^3$, while for higher temperatures such as T = 6 MeV it is convergent at higher densities up to 0.2 $nucleons/\mathrm{fm}^3$.

$\boldsymbol{n}$	$\boldsymbol{b_n}$
$n = 1$	0.3535533905933
$n = 2$	-0.0049500897299
$n = 3$	$1.483857713 \times 10^{-4}$
$n = 4$	$-4.4256301 \times 10^{-6}$
$n = 5$	1.006362×10^{-7}
$n = 6$	-4.272×10^{-10}

Table 4.1: Numerical values of the coefficients $\boldsymbol{b_n}$ calculated for the ideal Fermi gas

4.3 EVALUATION OF $\langle e^{i\vec{K}.\vec{r}/2}\rangle$ AND $\langle e^{-i\vec{k}.\vec{r}}\rangle$AT ABSOLUTE ZERO TEMPERATURE

According to Bose-Einstein condensation phenomenon, at absolute zero temperature bosons tend to accumulate in the lowest possible energy state, and they have zero momentum $\hbar\vec{K} = 0$ [6, 29]. Therefore

$$\langle e^{i\vec{K}.\vec{r}/2}\rangle = 1 \tag{4.41}$$

At absolute zero temperature, fermions can occupy the lowest distinct energy states up to the Fermi energy ε_F which is defined previously at the beginning of this chapter.

In this case, the Fermi-Dirac distribution function is defined as

$$f_{FD} = \begin{cases} 1, & \varepsilon \leq \varepsilon_F \\ 0, & \varepsilon > \varepsilon_F \end{cases} \tag{4.42}$$

Substituting equations (4.42) in equation (4.10) and (4.11), we get

$$\mathcal{A} = 4\left(\frac{L}{2\pi}\right)^3 \int_0^{k_F} d^3k \tag{4.43}$$

$$\langle e^{-i\vec{k}.\vec{r}}\rangle = \frac{4}{\mathcal{A}}\left(\frac{L}{2\pi}\right)^3 \int_0^{k_F} e^{-i\vec{k}.\vec{r}}\, d^3k \tag{4.44}$$

Where $\mathcal{A}$ is the total number of nucleons, and k_F is the Fermi wave vector corresponding to the Fermi energy ε_F. To evaluate this integral and without any loss of generality, we will assume that the vector $\vec{r}$ in equation (4.44) is directed along the z-axis, and so we have $e^{-i\vec{k}.\vec{r}} = e^{-ikrcos\theta}$. In spherical coordinates, the volume element in equations (4.43) and (4.44) is $d^3k = k^2 sin\theta dk d\theta d\varphi$ in k-space. Using spherical coordinates we can evaluate the integral analytically, and the result will be

$$\langle e^{-i\vec{k}.\vec{r}} \rangle = \frac{3\cos(k_F r)}{k_F^2 r^2} - \frac{3\sin(k_F r)}{k_F^3 r^3} \tag{4.45}$$

It is left to find k_F. To do so, let us recall that the total number of nucleons $\mathcal{A}$ is given by

$$\mathcal{A} = 4\left(\frac{L}{2\pi}\right)^3 \int_0^{k_F} d^3k = 4\left(\frac{L}{2\pi}\right)^3 \frac{4}{3}\pi k_F^3 \tag{4.46}$$

But the number density of nucleons inside the cubic volume L^3 is given by

$$\rho = \frac{\mathcal{A}}{L^3} \tag{4.47}$$

Substituting equation (4.46) in equation (4.47) and solving for k_F

$$k_F = \left(\frac{3\pi^2\rho}{2}\right)^{1/3} \tag{4.48}$$

Substituting the result of equation (4.48) in equation (4.45) we get

$$\langle e^{-i\vec{k}.\vec{r}} \rangle = \frac{3\cos\left(\left(\frac{3\pi^2\rho}{2}\right)^{1/3} r\right)}{\left(\frac{3\pi^2\rho}{2}\right)^{2/3} r^2} - \frac{3\sin\left(\left(\frac{3\pi^2\rho}{2}\right)^{1/3} r\right)}{\left(\frac{3\pi^2\rho}{2}\right) r^3} \tag{4.49}$$

Using equation (4.41) we get

$$\langle e^{i\vec{Q}.\vec{r}} \rangle = \langle e^{i\vec{K}.\vec{r}/2} \rangle \langle e^{-i\vec{k}.\vec{r}} \rangle = \frac{3\cos\left(\left(\frac{3\pi^2\rho}{2}\right)^{1/3} r\right)}{\left(\frac{3\pi^2\rho}{2}\right)^{2/3} r^2} - \frac{3\sin\left(\left(\frac{3\pi^2\rho}{2}\right)^{1/3} r\right)}{\left(\frac{3\pi^2\rho}{2}\right) r^3} \tag{4.50}$$

CHAPTER 5

USING GAUSSIAN POTENTIAL AND THE ISOLATED DEUTERON WAVEFUNCTION.

The binding energy in equation (4.1) is evaluated in general for any form of potential. In our recent work we use a Gaussian potential which is physically more reasonable.

We can write the Gaussian potential as follows

$$V(r) = -V_0 e^{-r^2/a^2} \tag{5.1}$$

Where V_0 and a are the depth and the range of the potential respectively. The values of V_0 and a can be determined using the variational principle and they can be adjusted to give the experimental value for the binding energy of the deuteron as we will do later in this chapter.

The trial wavefunction for the relative motion between the proton and the neutron inside the nucleus should be suitable to the form of the potential. So a Gaussian wavefunction is used as follows

$$g(r) = Ae^{-\eta r^2} \tag{5.2}$$

Where η is Gaussian parameter and A is the normalization constant $A^2 = \left(\frac{2\eta}{\pi}\right)^{\frac{3}{2}}$

5.1 FINDING THE VALUES OF THE PARAMETERS $V_{0t}, V_{0s}, a_t, \ a_s, \eta$

In this chapter we will use the isolated deuteron wavefunction so the parameter η is constant and does not vary with changing the vapor density.

For the ground state of the deuteron, the Hamiltonian can be expressed with the triplet interaction in the following equation

$$h_{12} = T + V_t = -\frac{(\hbar c)^2}{2\mu c^2}\nabla^2 - V_{0t} e^{-r^2/a_t{}^2} \tag{5.3}$$

Where μ is the reduced mass.

The value of a_t is allowed to vary in the range $1.5 > a_t > 3\ fm$ [30], and we choose a_t to be 2.05 fm as in [31]. V_{0t} and η can be adjusted to give us the correct binding energy of the deuteron and to ensure that it is a minimum. The energy for the isolated deuteron (which is the negative of the binding energy) is

$$E = < g|h_{12}|g > = \frac{3(\hbar c)^2\eta}{mc^2} - V_{0t}\left(\frac{2\eta}{2\eta + 1/a_t{}^2}\right)^{3/2} \tag{5.4}$$

Where g is the trial wavefunction defined in equation (5.2)

Using the variational principle we derive the energy in equation (5.4) with respect to η and by solving the equations $\frac{dE}{d\eta} = 0$ and $E(\eta)$ = -2.2 MeV, we find that the values of η and V_{0t} which minimize this energy are $V_{0t} = 47.7$ MeV and $\eta = 0.0936 fm^{-2}$. The energy $E(\eta)$ as a function of η is displayed in the figure (5.1).

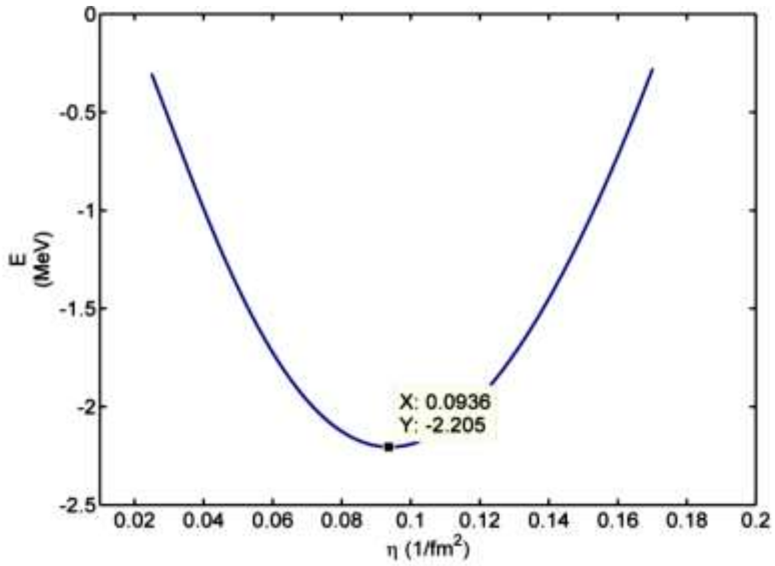

Figure 5.1: The energy of the isolated deuteron as a function of $\boldsymbol{\eta}$.
The value of $\boldsymbol{\eta}$ doesn't depend on the density $\boldsymbol{\rho}$

If we compare our results with the results in other references we find, for example, that in [32] the values of the depth and the range are $V_{0t} = 46.8$ MeV , $a_t = 1.94$ and the corresponding η equals $0.093 fm^{-2}$ which is close to ours, and the calculated binding energy in [32] is 2.133 MeV.

As we mentioned earlier in chapter 3, for the deuteron immersed in a vapor of nucleons, nucleons inside the deuteron interact with each other via the triplet interaction only, whereas at the same time they interact with the free nucleons via the triplet and singlet interactions and we should take this into account when we write the potential. So, in addition to the triplet potential we have the singlet potential with depth V_{0s} and range a_s

$$V_s(r) = -V_{0s} e^{-r^2/{a_s}^2} \tag{5.5}$$

The singlet state is not bound, it has a negative binding energy which is about 0.06 MeV [33], it exists only for a very short time in collisions. For the singlet state the range is 2.4

fm [31] and we can adjust the depth of the well until we have only one eigenstate with energy 0.06 MeV. We found that the required depth for the singlet interaction is 28.3 MeV. For the square well potential used in [10], the depth for the triplet interaction is 35 MeV and the range is 2.05 fm. While for the singlet interaction for the same potential, the depth is16 MeV and its range equals 2.4 fm.

Now, we have found all parameters we need to find the Mott density for the deuteron using Gaussian potential with a range (a_t) of $2.05\,\mathrm{fm}$ and depth (V_{0t}) $47.7\,\mathrm{MeV}$ for triplet interaction, and a range (a_s) of 2.4 fm and depth (V_{0s}) 28.3 MeV for the singlet interaction.

And the value for the parameter η used in the isolated deuteron wavefunction is $0.0936 fm^{-2}$.

In this case where the wavefunction is not allowed to vary with the vapor density we will use the same value of η ($0.0936 fm^{-2}$) at all densities. And we will use the formula for the energy in equation (3.23)

$$\langle\Psi_{\mathrm{tot}}|\mathcal{H}|\Psi_{\mathrm{tot}}\rangle = -B_0 - \frac{3}{8}\rho.\,\mathrm{J}^*.\,\mathrm{J}_{2\mathrm{t}} - \frac{1}{8}\rho.\,\mathrm{J}^*.\,\mathrm{J}_{2\mathrm{s}} = -B(\rho)$$

Where B_0 in this case is constant and its value equals $2.2\,MeV$ which is the binding energy for the isolated deuteron in the ground state, and ρ is the total vapor density.

$\mathrm{J}, \mathrm{J}_{2\mathrm{t}}, \mathrm{J}_{2s}$are the integrals defined in equations (3.9), (3.18), and (3.19). These integrals depend on the average quantities $\langle e^{-i\vec{k}.\vec{r}}\rangle$ and $\langle e^{i\vec{K}.\vec{r}/2}\rangle$ where these quantities depend on the density of the free nucleons (ρ_{free}). So for a certain density ρ we make an iteration over the values of ρ_{free} which are less than or equal to the total density ρ ($\rho_{free} \leq \rho$), and the iteration will stop at a certain value of ρ_{free}.

At this value of ρ_{free} the density of free (unbound) nucleons ρ_{free} plus the density of bound nucleons ρ_d, which join together to give the deuteron, should be very close to the value of the total density ρ we start with (with a relative error (r) less than 0.01). In other words, the iteration over ρ_{free} stops when the value $r = \frac{\rho_{tot}-\rho}{\rho}$ is not larger than 0.01, where $\rho_{tot} = \rho_{free} + 2\rho_d$. The factor of 2 in the equation reflects the fact that the deuteron contains 2 nucleons.

ρ_d is defined in equation (4.37)

$$\rho_d = \frac{\mathcal{A}'}{L^3} = \frac{3}{(2\pi)^3}\int \acute{f}_d d^3K$$

where $\acute{f}_d = \frac{1}{e^{\beta(\acute{\varepsilon}-\mu'-B(\rho))}-1}$, and $\mu' = 2\mu$

At this value of ρ_{free} which achieves self consistency for ρ_{tot}, we use equation (3.24)

$$B(\rho) = -\langle\Psi_{\mathrm{tot}}|\mathcal{H}|\Psi_{\mathrm{tot}}\rangle = B_0 + \frac{3}{8}\rho.\mathrm{J}^*.\mathrm{J}_{2\mathrm{t}} + \frac{1}{8}\rho.\mathrm{J}^*.\mathrm{J}_{2\mathrm{s}}$$

to calculate the deuteron binding energy. To achieve the self consistency for binding energy another iterative operation is performed. The iteration of the binding energy should stop when the difference between two successive values for binding energy does not exceed 0.001 MeV as we discussed in section 4.2.

5.2 BINDING ENERGY RESULTS USING ISOLATED DEUTERON WAVEFUNCTION

In this section, we will use the values of V_{0t}, V_{0s}, a_t, a_s, and η obtained in section 5.1 above to investigate the binding energy when the deuteron wavefunction is not allowed to vary with the vapor density.

We will plot the binding energy of the deuteron in equation (3.24) versus the density ρ

$$B(\rho) = B_0 + \frac{3}{8}\rho . \mathrm{J}^* . \mathrm{J}_{2\mathrm{t}} + \frac{1}{8}\rho . \mathrm{J}^* . \mathrm{J}_{2\mathrm{s}}$$

Where $\mathrm{J}, \mathrm{J}_{2\mathrm{t}}$, and J_{2s} are defined in equations (3.9), (3.18), and (3.19) respectively.

But at first we will show why we take the first 7 terms in equations (4.31) and (4.33). In the figure (5.2) below we plot the binding energy of the deuteron as a function of the vapor density (ρ) at $T = 20\ MeV$ using different number of terms.

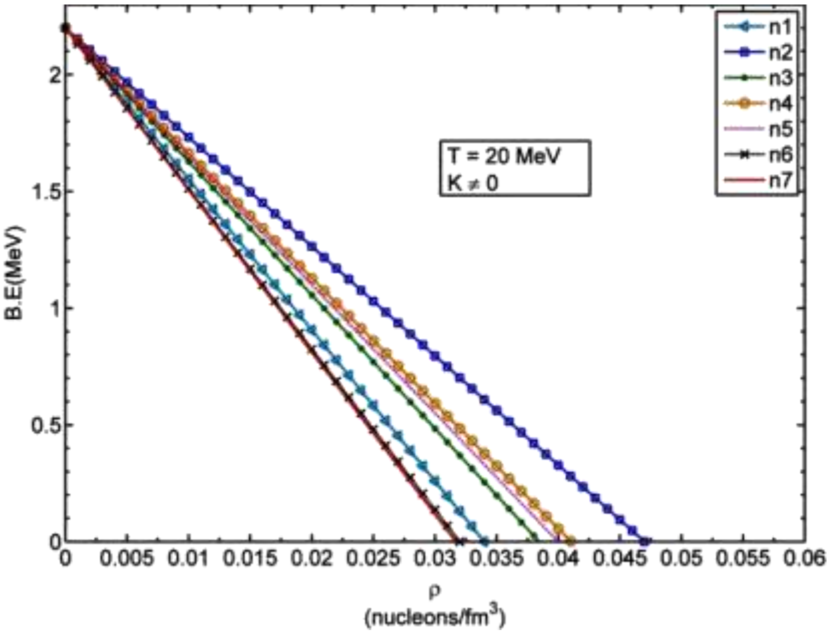

Figure 5.2: Deuteron binding energy as a function of $\boldsymbol{\rho}$ for different number of terms in the high temperature expansion in equations **(4.31)** and **(4.33)**.

It is clear that when we take the first six terms (the black line) and the first seven terms (the red line) in the expansion we almost get the same curves, which indicates that it is enough to take the first seven terms to achieve the convergence of the series. But for less terms the series is divergent.

In the following figures (5.3 - 5.5) we plot the results obtained in the present work using Gaussian potential and the results obtained in [10] using square well potential for nonzero CM momentum (K≠0) and for zero CM momentum (K = 0). In these plots the wavefunction is not allowed to vary with the vapor density.

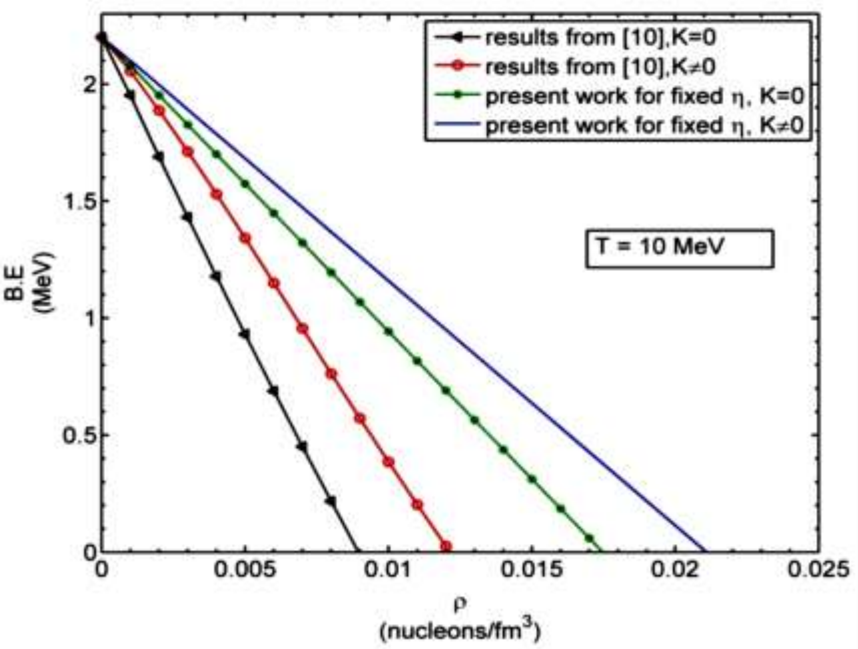

Figure 5.3: Deuteron binding energy at **T=10 MeV**. The results from [10] for a square well potential are given by the red line for K≠0 and the black line for K=0. The present results for a Gaussian potential are shown by the blue line for the case of K≠0 and the green line for K=0. For both potentials, the wavefunction is not allowed to vary with the vapor density.

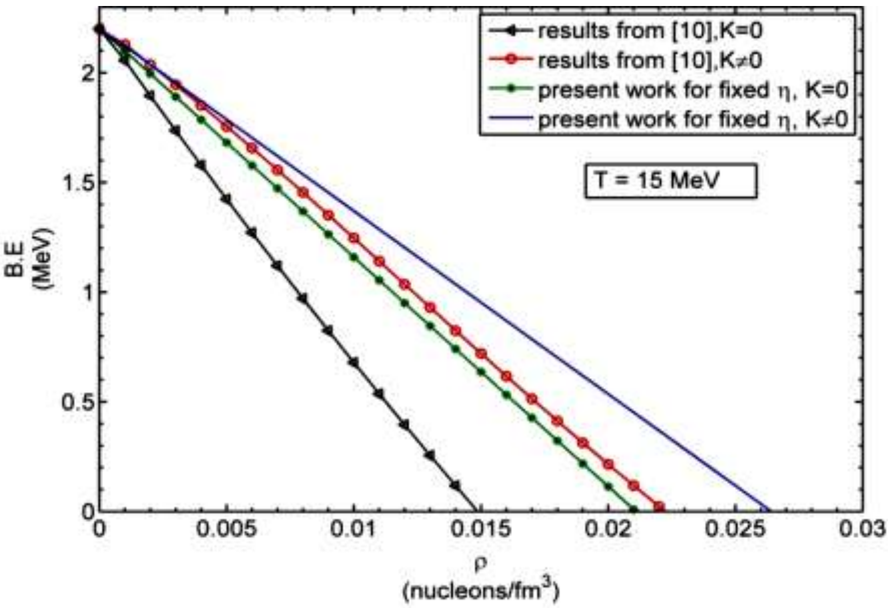

Figure 5.4: Deuteron binding energy at **T=15 MeV**. The results from [10] for a square well potential are given by the red line for K≠0 and the black line for K=0. The present results for a Gaussian potential are shown by the blue line for the case of K≠0 and the green line for K=0. For both potentials, the wavefunction is not allowed to vary with the vapor density.

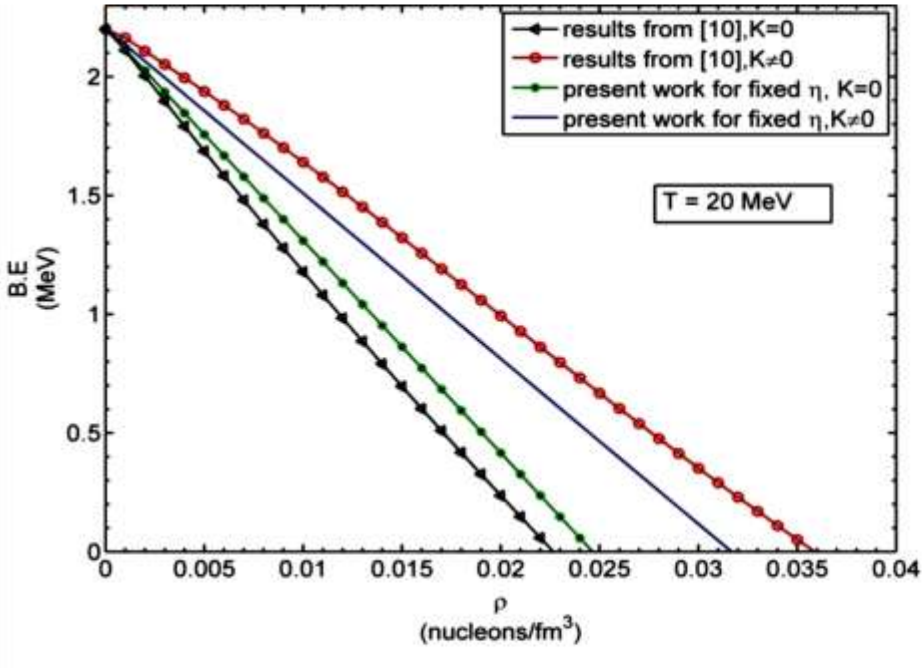

Figure 5.5: Deuteron binding energy at **T=20 MeV**. The results from [10] for a square well potential are given by the red line for K≠0 and the black line for K=0. The present results for a Gaussian potential are shown by the blue line for the case of K≠0 and the green line for K=0. For both potentials, the wavefunction is not allowed to vary with the vapor density.

From the figures above (5.3-5.5), we can see that the behavior of binding energy in all cases is the same; it decreases linearly with the vapor density. In the present work, the deuteron immersed in vapor of nucleons has higher Mott density at low temperatures and it can survive more before it dissolves. But at high temperatures it has lower Mott density than in [10] when K.$\neq$ 0.

When we compare the results for K=0 with K$\neq$0, it is clear that the Mott density for the zero CM momentum case where the deuteron is assumed to be at rest is less than the case where the CM momentum does not equal zero. This is reasonable, because when the CM momentum of the deuteron equals zero the momenta of the nucleons inside this deuteron also equal zero which are the minimum and the effect of Pauli blocking will be large so the deuteron will dissolve.

In general, we can notice from the figures that the difference in the Mott density between the two cases; the square well and Gaussian potentials decreases as the temperature increases.

To see what happens at absolute zero temperature (T = 0 MeV), we plot the deuteron binding energy in figure (5.6).

According to Bose-Einstein condensation phenomenon, at absolute zero temperature (T = 0) bosons tend to accumulate in the lowest possible energy state, and they have zero momentum (K = 0) which is already included in the calculations and in the matlab code. For this reason, there is no need for the case of nonzero CM momentum.

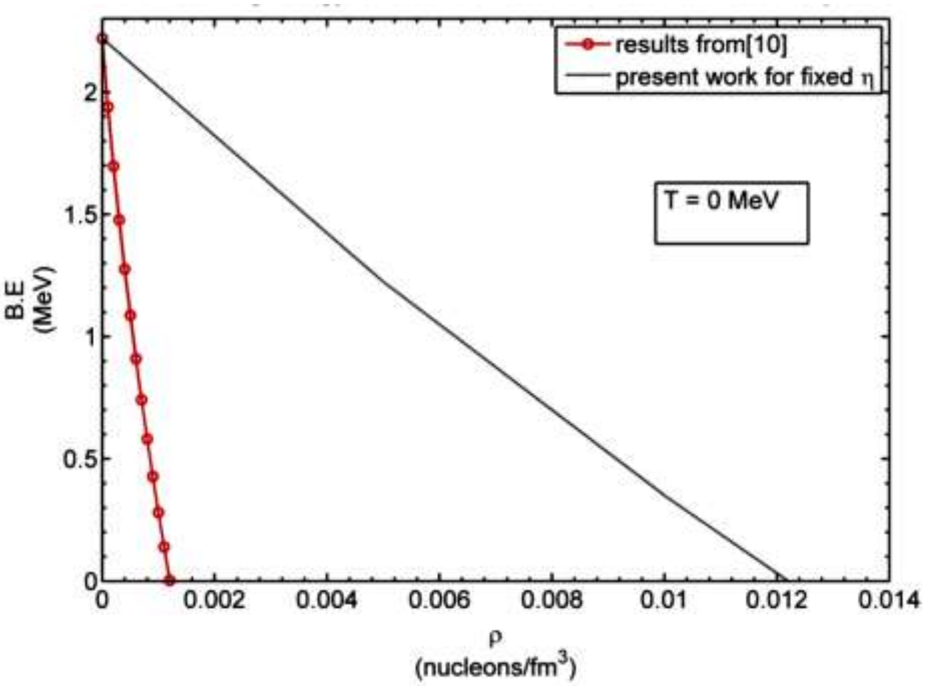

Figure 5.6: Deuteron binding energy at **T=0 MeV**. The results from [10] for a square well potential are given by the red line. The present results for a Gaussian potential are shown by the black line. For both potentials, the wavefunction is not allowed to vary with the vapor density.

The behavior of binding energy at T = 0 MeV in figure (5.6) is similar to its behavior at higher temperatures. The Mott density for present work is higher than the result reported in [10].

In chapter 6 we will study the behavior of the deuteron binding energy when its wavefunction is allowed to vary with vapor density.

CHAPTER 6

USING THE VARIATIONAL PRINCIPLE TO DETERMINE THE GAUSSIAN WAVEFUNCTION.

In this chapter we will find the binding energy for the deuteron immersed in a vapor of nucleons when the Gaussian wavefunction is allowed to vary with the density of the vapor, that is different from that of an isolated deuteron.

6.1 FINDING THE VALUE OF η USING THE VARIATIONAL PRINCIPLE

The value of η in this chapter is not fixed so we will use the variational principle to minimize the deuteron energy with respect to η at each density. We will use the same formula for energy in equation (3.23) with V_{0t}, V_{0s}, a_t, a_s obtained in section 5.1

$$E = \langle \Psi_{\mathrm{tot}} | \mathcal{H} | \Psi_{\mathrm{tot}} \rangle = -B_0 - \frac{3}{8}\rho . \mathrm{J}^* . \mathrm{J}_{2\mathrm{t}} - \frac{1}{8}\rho . \mathrm{J}^* . \mathrm{J}_{2\mathrm{s}} \tag{6.1}$$

Where ρ is the total density of the vapor, $\mathrm{J}, \mathrm{J}_{2\mathrm{t}}, \mathrm{J}_{2\mathrm{s}}$ are the integrals defined in equations (3.9), (3.18), and (3.19).

Since $\mathrm{J}, \mathrm{J}_{2\mathrm{t}}, \mathrm{J}_{2\mathrm{s}}$ depend on the density of the free nucleons (ρ_{free}), the value of ρ_{free} which achieves the self consistency for the total density can be found using the same method explained in section (5.1).The value of η in this chapter is not fixed and hence the initial value for the binding energy(B_0) is not a constant , it depends on the parameter η. From equation (5.4)

$$-B_0 =< g|h_{12}|g > = \frac{3(\hbar c)^2\eta}{mc^2} - V_{0t}\left(\frac{2\eta}{2\eta+1/a_t{}^2}\right)^{3/2}$$

At the value of ρ_{free} which achieves self consistency for ρ_{tot} defined in section 5.1, we minimize the deuteron energy with respect to η to find its minimum value. We do this by plotting the deuteron energy in equation (6.1) versus η at each density ρ and find the minimum value for energy.

To achieve self-consistency for the energy another iterative operation is performed. The iteration for energy should stop when the difference between two successive values for energy does not exceed 0.001 MeV as we said in section 4.2.

For each ρ there is a certain value of η where the deuteron energy has its minimum (or the binding energy has its maximum). Which indicates that the value of η varies with the density ρ. We illustrate this by plotting the energy in equation (6.1) as a function of η for two different densities; $\rho = 0.006\, nucleons/fm^3$and $\rho = 0.01\, nucleons/fm^3$at the same temperature T= 10 MeV with K = 0. The results are presented in the figures(6.1)and (6.2).

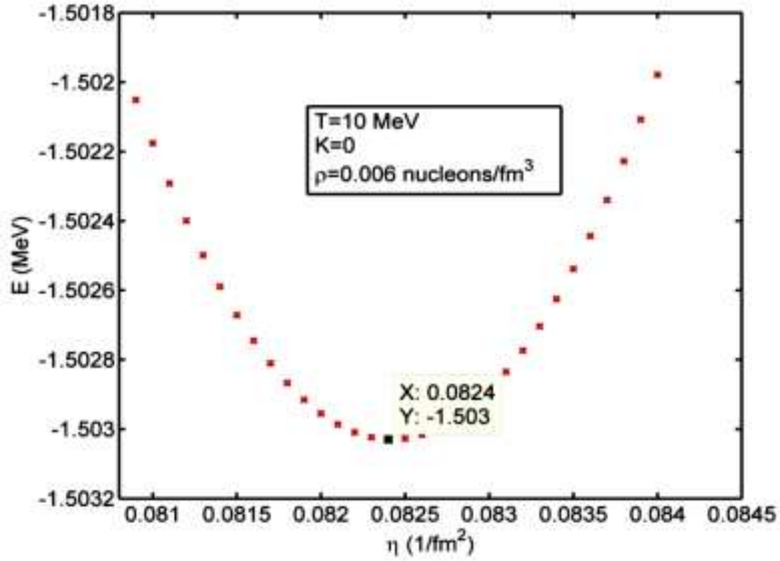

Figure 6.1: Deuteron energy as a function of $\boldsymbol{\eta}$ at $\boldsymbol{\rho = 0.006}$ nucleons/fm^3, T = 10 MeV, K = 0.

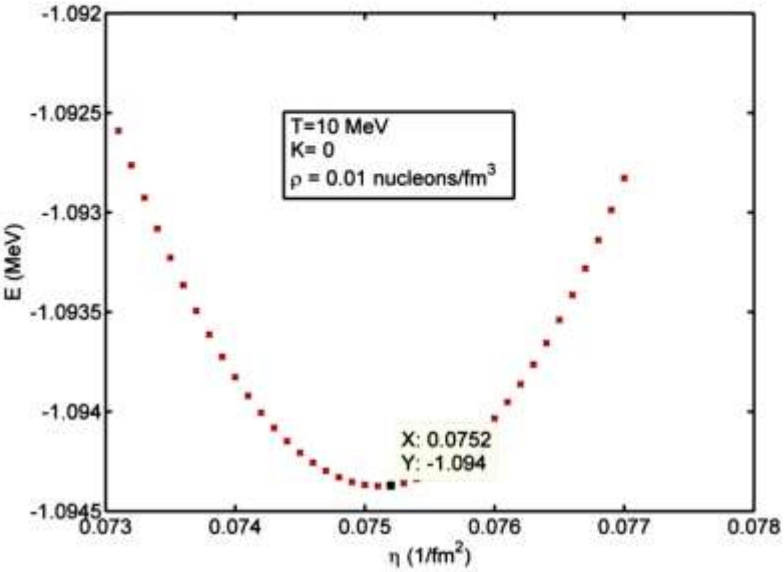

Figure 6.2: Deuteron energy as a function of $\boldsymbol{\eta}$ at $\boldsymbol{\rho = 0.01}$ nucleon/fm^3, T = 10 MeV, K = 0.

From the figures (6.1) and (6.2) above, it is obvious that the value of η which minimizes the energy depends on the vapor density. At $\rho = 0.006\ nucleons/fm^3$ the energy has its minimum $E = -1.503\ MeV$ when $\eta = 0.0824\ fm^{-2}$, while at $\rho = 0.01\ nucleons/fm^3$ the minimum energy $E = -1.094 MeV$ occurs when $\eta = 0.0752\ fm^{-2}$. The two values were obtained at the same temperature $T = 10\ MeV$ and with zero CM momentum.

The same method is followed for other densities to find the value of η and its corresponding minimum energy. In the table (6.1) we summarize the results obtained for different densities at $T = 10\ MeV$ with $K = 0$.

Total density $\rho\ (nucleons/fm^3)$	$\eta(fm^{-2})$	The corresponding minimum energy E (MeV)
0 (isolated deuteron)	0.0936	- 2.2
0.001	0.092	- 2.080
0.002	0.090	- 1.959
0.003	0.0880	- 1.840
0.005	0.0840	- 1.612
0.006	0.0824	- 1.503
0.007	0.0810	- 1.396
0.008	0.0790	- 1.293
0.01	0.0752	- 1.094
0.011	0.0730	- 0.9992
0.012	0.0720	- 0.9069
0.013	0.0700	- 0.8174
0.015	0.0660	- 0.6464
0.016	0.0650	- 0.5649
0.018	0.0610	- 0.4099
0.019	0.0600	- 0.3362
0.022	0.0542	- 0.1308
0.023	0.0536	- 0.06758
0.024	0.0510	- 0.00672
0.025	0.0492	0.05161

Table 6.1: the values of $\boldsymbol{\eta}$ and their corresponding minimum energy for each $\boldsymbol{\rho}$ at T = 10 MeV with K = 0.

From the table (6.1) we can notice that the value of η decreases as the vapor density increases.

Each value of η is found by plotting the energy of the deuteron versus the parameter η at each density ρ as shown in figures (6.1) and (6.2) for $\rho = 0.006\ nucleons/fm^3$ and for $\rho = 0.006\ nucleons/fm^3$,, then the value of η which minimizes the energy is recorded with its minimum energy. After that we plot the negative of the third column in the table (the binding energy) versus the total density ρ to find the Mott density; where the deuteron dissolves and becomes unbound.

6.2 BINDING ENERGY RESULTS WHEN THE WAVEFUNCTION IS ALLOWED TO VARY WITH THE VAPOR DENSITY

The results in table (6.1) at T = 10 MeV beside the results at other temperatures are presented in the figures (6.3-6.5) below. In each figure, we plot the deuteron binding energy versus the vapor density (ρ) for zero CM momentum (K = 0) and for nonzero CM momentum (K≠0). We compare the results for the case where the wavefunction is not allowed to vary (η is fixed) with the case where the wavefunction is allowed to vary (η is variable).

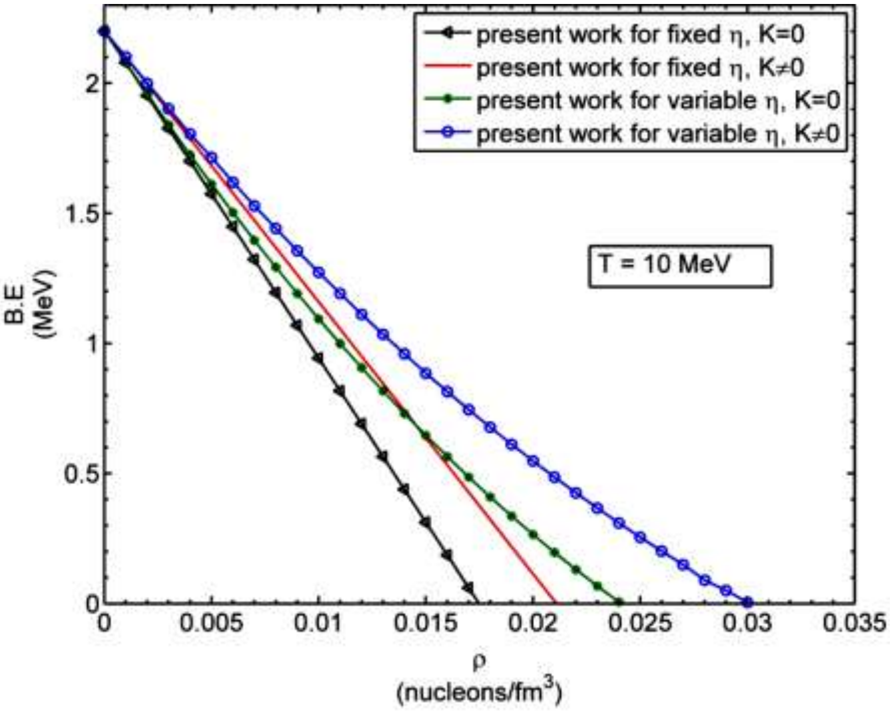

Figure 6.3: Deuteron binding energy at **T=10 MeV**. The present results for fixed η are given by the red line for K≠0 and the black line for K=0. When η is variable, the present results are shown by the blue line for the case of K≠0 and the green line for K=0.

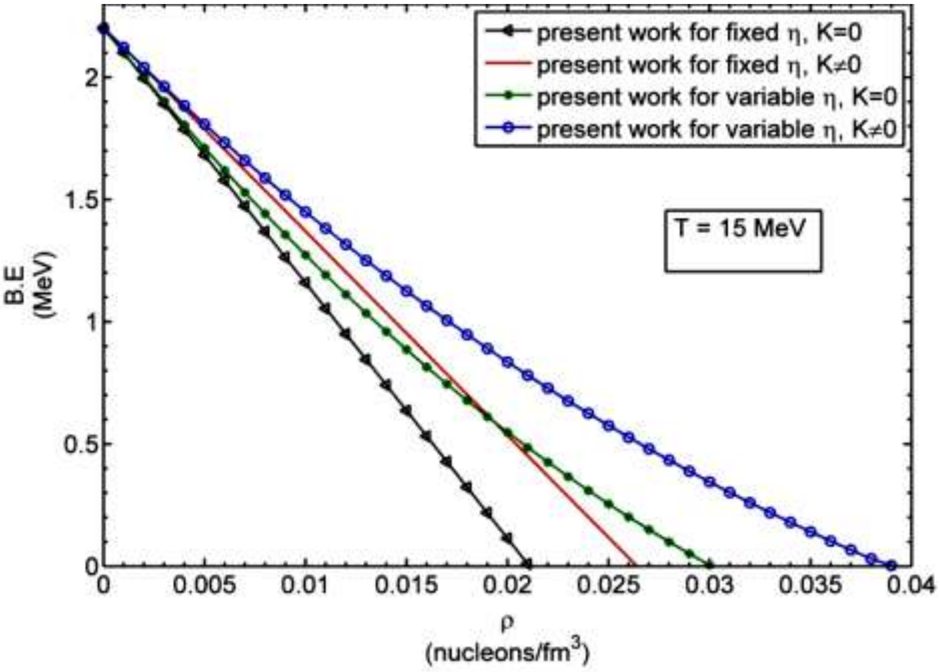

Figure 6.4: Deuteron binding energy at **T=15 MeV**. The present results for fixed η are given by the red line for K≠0 and the black line for K=0. When η is variable, the present results are shown by the blue line for the case of K≠0 and the green line for K=0.

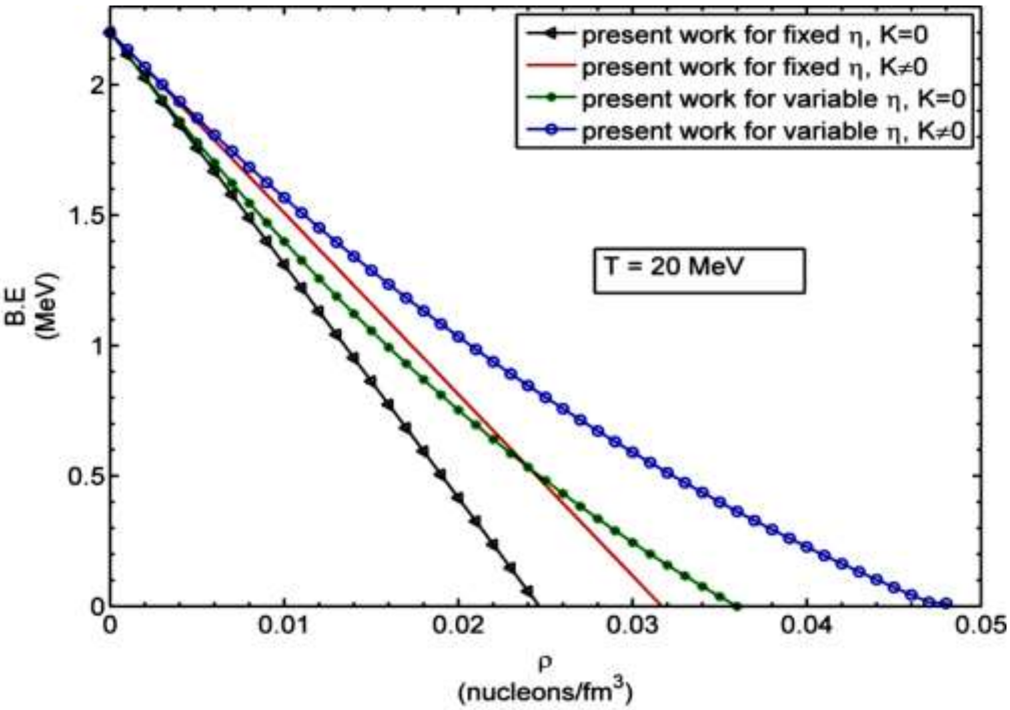

Figure 6.5: Deuteron binding energy at **T=20 MeV**. The present results for fixed η are given by the red line for K≠0 and the black line for K=0. When η is variable, the present results are shown by the blue line for the case of K≠0 and the green line for K=0.

From the three figures (6.3 - 6.5) above, we can see the effect of changing the parameter η with the vapor density on the shape of the binding energy curve. For this case, the binding energy decreases nonlinearly as the density increases, while for isolated deuteron where η is fixed, the binding energy decreases linearly. We can also notice that the deuteron will survive more when the Gaussian wavefunction varies with the vapor density. The decrease in the Mott density for the zero CM momentum case is expected for the same reasons explained previously in chapter 5.

Now, we will make a plot for the deuteron binding energy as a function of the vapor density at absolute zero temperature (T = 0 MeV).

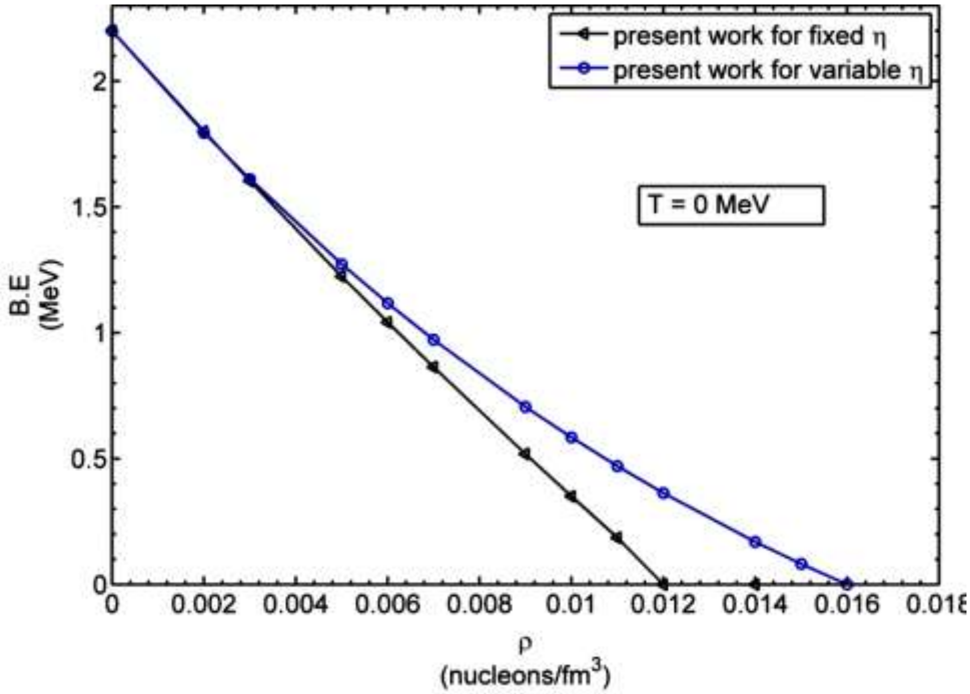

Figure 6.6: Deuteron binding energy at **T = 0 MeV**. The black line shows the present results when η is fixed and the blue line for the present results when η is variable.

The same behavior for deuteron energy can be noticed from the figure (6.6) at T = 0 MeV. The Mott density is larger for the case at which the deuteron wavefunction is allowed to vary with the vapor density.

There is no separate curve for the zero CM momentum case (K=0), because at absolute zero temperature all bosons tend to accumulate in the lowest possible energy state, and they have zero momentum (K = 0).

Chapter 7

RESULTS AND CONCLUSION

In this chapter, we will summarize our results for the Mott density, where the deuteron dissolves and its binding energy equals zero, using Gaussian potential and compare them with the results reported in [10] by using the square well potential.

The new thing in our work is including the effect of the vapor density in the wavefunction. So we will compare our results for the case where the wavefunction is not allowed to vary with the vapor density with the other case where the wavefunction is allowed to vary with the vapor density.

The values for Mott density for all cases can be summarized in the following table (7.1).

Temperature (MeV)	CM momentum (K)	Results obtained in [10] (Square well potential)	Results of the present work for fixed η (Gaussian potential)	Results of the present work when η is notfixed (Gaussian potential)
0		0.0012	0.012	0.016
10	$K = 0$	0.009	0.017	0.024
	$K \neq 0$	0.012	0.021	0.03
15	$K = 0$	0.015	0.021	0.03
	$K \neq 0$	0.022	0.026	0.039
20	$K = 0$	0.023	0.025	0.036
	$K \neq 0$	0.036	0.031	0.048

Table 7.1: Mott densities for the deuteron at different temperatures obtained in the present work, along with those obtained in [10].

From the table, we can notice that for all cases the Mott density increases as the temperature increases, which means that the deuteron survives more at higher temperatures.

By looking at the cases of K = 0 and K ≠ 0 at each temperature, it is obvious that the deuteron with nonzero center of mass momentum can survive more before it dissolves to its constituents. The reason behind this is that the Pauli blocking has more effect in the case of zero center of mass momentum.

The comparison between the results reported by Abdul-Rahman, Alstaty, and Jaqaman in [10] sing square well potential and the present work using Gaussian potential reflects the effect of the potential shape on the deuteron energy. We observe that at low temperatures the deuteron has much higher Mott density than in [10]. But this difference in the Mott density between the two studies decreases as the temperature increases.

The effect of the dependence of the wavefunction on the vapor density can be noticed in the last column in the table. Obviously, the Mott density for the deuteron is higher than its value for other cases where the wavefunction is not affected by the density of the vapor.

In this study, we used Gaussian potential and the variational principle to investigate the effect of the surrounding vapor on the deuteron stability and to find the Mott density . We considered that the vapor consists of nucleons and deuterons only. In the future, we can include other clusters present in the vapor, such as helion (^{3}He) and alpha (^{4}He) clusters.

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Appendix A

```
% BE when the wavefunction is allowed to vary with the vapor density
hc=197.33;%blank constant
mc2=940;%nucleon mass
Mc2=1876.12;%deuteron mass
gamat=0.238;%fm^-2
gamas=0.1736;

Ut=47.7;%MeV potential depth
Us=28.3;%MeV potential depth


b1=0.3535533905933;
b2=-0.0049500897299;
b3=1.483857713*10^(-4);
b4=-4.4256301*10^(-6);
b5=1.006362*10^(-7);
b6=-4.272*10^(-10);

KbT=15;%Mev
beeta=1/KbT;



rho=0.04
foreta=0.045:0.001:0.048

           I=rho

rhofree = 0.00000005;

while(rhofree<rho)


eta

 A=(2*eta/pi)^(3/4);%Normalization constant for the gaussianwavefunction

bindenerg=-1.2;%initial value
lam3=(2*pi*(hc^2)/(mc2*KbT))^(1.5);
eita=(rhofree*lam3)/4;


muu=(log(eita)+
b1.*eita+b2*(eita.^2)+b3*(eita.^3)+b4*(eita.^4)+b5*(eita.^5)+b6*(
eita.^6));%muu/KT for nucleons
expmu=(eita*exp(b1*eita+b2*(eita^2)+b3*(eita^3)+b4*(eita^4)+b5*(e
ita^5)+b6*(eita^6)));%exp(muu/kt)
```

```
DD=0;

while (1)%iteration to find the binding energy(negative value)



expBofrho=exp(bindenerg);


symsr;
Expectfermion=((exp((-mc2*KbT*(r^2))/(2*(hc^2))))-(expmu)*exp((-
mc2*KbT*(r^2))/(4*(hc^2)))*(1/(2*(2^0.5)))    +    (expmu^2)*exp((-
mc2*KbT*(r^2))/(6*(hc^2)))*(1/(3*(3^0.5)))-(expmu^3)*exp((-
mc2*KbT*(r^2))/(8*(hc^2)))*(1/(4*(4^0.5)))+(expmu^4)*exp((-
mc2*KbT*(r^2))/(10*(hc^2)))*(1/(5*(5^0.5)))-(expmu^5)*exp((-
mc2*KbT*(r^2))/(12*(hc^2)))*(1/(6*(6^0.5)))      +(expmu^6)*exp((-
mc2*KbT*(r^2))/(14*(hc^2)))*(1/(7*(7^0.5))))/(1-
((expmu)/(2*(2^0.5)))+((expmu^2)/(3*(3^0.5)))-
((expmu^3)/(4*(4^0.5)))+((expmu^4)/(5*(5^0.5)))-
((expmu^5)/(6*(6^0.5)))+((expmu^6)/(7*(7^0.5))));


Expectboson=(exp((-mc2*KbT*(r^2))/(4*(hc^2)))
+((expmu^2)*(expBofrho))*exp((-
mc2*KbT*(r^2))/(8*(hc^2)))*(1/(2*(2^0.5)))
+((expmu^4)*(expBofrho^2))*exp((-
mc2*KbT*(r^2))/(12*(hc^2)))*(1/(3*(3^0.5)))+((expmu^6)*(expBofrho
^3))*exp((-mc2*KbT*(r^2))/(16*(hc^2)))*(1/(4*(4^0.5)))
+((expmu^8)*(expBofrho^4))*exp((-
mc2*KbT*(r^2))/(20*(hc^2)))*(1/(5*(5^0.5)))
+((expmu^10)*(expBofrho^5))*exp((-
mc2*KbT*(r^2))/(24*(hc^2)))*(1/(6*(6^0.5))+((expmu^12)*(expBofrho
^6))*exp((-
mc2*KbT*(r^2))/(28*(hc^2)))*(1/(7*(7^0.5)))/(1+((expmu^2)*(expBof
rho))*(1/(2*(2^0.5)))+((expmu^4)*(expBofrho^2))*(1/(3*(3^0.5)))+(
(expmu^6)*(expBofrho^3))*(1/(4*(4^0.5)))+((expmu^8)*(expBofrho^4)
)*(1/(5*(5^0.5)))+((expmu^10)*(expBofrho^5))*(1/(6*(6^0.5))))+((e
xpmu^12)*(expBofrho^6))*(1/(7*(7^0.5)))));

f=inline(exp(-
1*eta*r^2)*char(Expectfermion)*char(Expectboson),'r');
f2=inline(exp(-
1*(eta+gamat)*r^2)*char(Expectfermion)*char(Expectboson),'r');
f3=inline(exp(-
1*(eta+gamas)*r^2)*char(Expectfermion)*char(Expectboson),'r');


J=4*pi*A*quadl(f,0,1000000);
J2t=-Ut*4*pi*A*quadl(f2,0,1000000);
J2s=-Us*4*pi*A*quadl(f3,0,1000000);
```

```
bindenerg=                                  (((3*(hc)^2)/mc2)*eta-
(Ut*(2*beta/(2*beta+gamat))^(3/2))-(0.375*conj(J)*J2t*rho)-
(0.125*conj(J)*J2s*rho))

DX= abs(bindenerg-DD)


if (DX <=0.001)

break

end
DD= bindenerg


end


%%deuteron

KE2=(hc^2/(2*Mc2));
zzz=exp(-1*beeta*bindenerg)*exp(-2*muu);
ff=(@(x)((x.^2)./((zzz*(exp(beeta*KE2*(x.^2))))-1)));

q2 = quadgk(ff,0,inf);
y2 =( 3*q2)/2*(3.14)^2;

rhodeuteron=2*y2

%total density
tot=2*y2+rhofree
rr=abs((tot-I)/I)

if (rr<0.01)



plot(eta,bindenerg)
holdon

break
else

rhofree=rhofree+0.00000005;

I=tot

end

end
end
```